JOURNAL OF GEOPHYSICAL RESEARCH, VOL. 91, NO. A2, PAGES 1519–1528, FEBRUARY 1, 1986

Shear Flow–Ballooning Instability as a Possible Mechanism for Hydromagnetic Fluctuations

ADOLFO F. VIÑAS

Laboratory for Extraterrestrial Physics, NASA Goddard Space Flight Center, Greenbelt, Maryland

THEODORE R. MADDEN

Massachusetts Institute of Technology, Cambridge

It is suggested that an MHD instability termed the "shear flow–ballooning instability," which unifies both the Kelvin-Helmholtz and the interchange ("ballooning") instabilities, can excite hydromagnetic waves in the inner magnetosphere. The stability analysis resembles studies of hydrodynamic flows, where the stabilizing factor is the gravitational buoyancy represented by the Brunt-Väisälä (or Rayleigh-Taylor) frequency $\Omega_g(r)$. Here the "magnetic buoyancy" due to the curvature of the field lines replaces the gravitational buoyancy and allows the derivation of the MHD analogue to $\Omega_g(r)$. Stability is then found to depend on a dimensionless quantity termed the magnetic Richardson number (similar to hydrodynamic) $Ri = [\Omega_g^2(r) + k_\parallel^2 C_a^2](1 + k_\parallel^2/k_\perp^2)/(dV_\phi/dr)^2$, representing the relative importance of gravitational, thermal, rotational, magnetic, and shear flow effects. Unstable MHD modes are found to be represented by Alfvén drift waves which are the hydromagnetic, and shear flow effects. Unstable MHD modes are found to be represented by Alfvén drift waves which are the hydromagnetic analogue to hydrodynamic gravity waves and like them are trapped in the shear zone. The study is applied to the plasmapause boundary, and the results indicate that low-frequency hydromagnetic pulsations (Pc 4–Pc 5) with typical wave periods between 123 and 428 s and wavelengths in the range of 5×10^3 to 17.2×10^3 km can be excited in such a region. The analysis can be extended to other shear flow boundaries such as the magnetopause.

1. INTRODUCTION

Considerable observational evidence exists which indicates that low-frequency hydromagnetic waves occur in the inner magnetosphere in association with geomagnetic storms [e.g., *Barfield and Coleman*, 1970; *Barfield and McPherron*, 1972; *Barfield et al.*, 1972; *Dwarkin et al.*, 1971; *Lanzerotti et al.*, 1974, 1975; *Lanzerotti and Fukunishi*, 1975; *Lanzerotti and Maclennan*, 1976]. Further evidence indicates that some of these fluctuations are associated with the outer regions of the plasmasphere (i.e., the plasmapause) where plasma elements are peeled off during periods of enhanced geomagnetic activity, particularly at the dusk sector [e.g., *Kikuchi*, 1971, 1976; *Kivelson*, 1976; *Lanzerotti et al.*, 1974, 1975; *Lanzerotti and Fukunishi*, 1975; *Lanzerotti and Maclennan*, 1976]. Various mechanisms have been suggested to explain the excitation of these hydromagnetic fluctuations. Among them the gradient drift and the Kelvin-Helmholtz (K-H) instabilities have emerged as the most probable sources for hydromagnetic waves in the magnetosphere (see reviews by *Lanzerotti and Southwood* [1979] and *Southwood and Hughes* [1983]).

The study of low-frequency gradient drift waves at the plasmapause has previously been investigated by *Kikuchi* [1971, 1976], *Hasegawa* [1971], *Hasegawa and Chen* [1974], *Patel* [1978], *Migliuolo and Patel* [1981], *Migliuolo* [1983], and *Southwood and Hughes* [1983], among others. Owing to the substantial inhomogeneities in density and temperature (i.e., sharp gradients) at the plasmapause and to the presence of hot particles, hydromagnetic drift waves of internal origin can be excited as a consequence of the particle drift motion in a magnetic field. These waves can propagate across the magnetic field and can couple on resonant field lines to yield shear Alfvén waves. Because of the presence of hot particles from the plasma sheet, the plasma pressure can sometimes be comparable to the magnetic pressure at the plasmapause, particularly during geomagnetic storms. This is an important effect, because it suggests that the plasma internal energy is likely to be a source of wave energy and that variations in the internal energy of the plasma can be associated with similar variations in the magnetic field, suggesting the compressive nature of the fluctuations. Owing to the presence of these hot particles an MHD ballooning or interchange instability can occur at the plasmapause. This instability, which is driven by variations in the thermal energy of the plasma in an unfavorable magnetic field curvature, has been previously investigated in both the space and plasma physics context by *Gold* [1959], *Sonnerup and Laird* [1963], *Chang et al.* [1965], *Liu* [1970], *Richmond* [1973], and *Coppi et al.* [1979], among others. A somewhat different approach to the interchange instability at the plasmapause has been investigated by *Lemaire* [1974, 1975, 1976] and *Lemaire and Kowalkowski* [1981]. They have suggested that the formation of the plasmapause is caused by a centrifugally-gravitationally driven interchange mechanism, although no indication of the type of wave modes excited is presented in their discussion.

On the other hand, the theory of the K-H shear flow instability has been intensively investigated by *Fejer* [1964], *Sen* [1964], *Southwood* [1968], *Gerwin* [1968], *Ong and Roderick* [1972], and *Miura and Pritchett* [1982], among others. However, most of the application of this instability to space plasma physics has been directed to the study of the stability of the magnetopause boundary, and very little has been done with respect to the plasmapause boundary. This instability is driven by the relative streaming of the plasma flow (i.e., velocity shear) where perturbations can grow at the expense of the kinetic energy of the flow, resulting in turbulence and mixing.

Paper number 5A8830.
0148-0227/86/005A-8830$05.00

Observations of the high electric field at low ionospheric altitudes at the edge of the diffuse aurora on field lines that map in the vicinity of the plasmapause suggest the presence of large electric (or velocity) shears during periods of enhanced activity [*Smiddy et al.*, 1977; *Maynard*, 1978; *Rich et al.*, 1980]. Such spatial variation in the electric field has been previously suggested by *Chappell* [1974] from plasma observations of detached regions during magnetically disturbed periods.

The purpose of this paper is to present a unified linear electromagnetic analysis of both the K-H (shear flow) instability and the ballooning (interchange) instability based on MHD theory. We extended the concepts of the Richardson instability of hydrodynamic flows into the hydromagnetic context by unifying both the shear flow K-H instability and the ballooning instability. An essential concept of this analysis is the role played by the magnetic buoyancy due to an effective gravity produced by the curvature of the field lines which provides the basic step by which both instabilities could be coupled. By virtue of the concept of magnetic buoyancy one can demonstrate from purely dynamical principles the criterion of instability of ballooning modes in a dipole field. In the past this condition has always been derived from energy principles. Although energy considerations give accurate global instability criteria, information associated with hydromagnetic waves and their properties cannot be obtained from this kind of approach. We will then apply the results of this analysis to the plasmapause to explain the excitation of hydromagnetic waves in that region, including the effect of the hot particles from the plasma sheet.

This unified treatment of the K-H and ballooning instabilities rests on a series of physical assumptions: (1) our analysis does not consider the "line-tying" effect at the foot of the field lines due to a finite conductor (such as the ionosphere) which would tend to stabilize the interchange mode; (2) since the analysis is based upon the MHD theory, kinetic effects are not included; (3) we neglect effects due to pressure anisotropy; and (4) the frozen-in law is assumed. *Chang et al.* [1965], *Liu* [1970], and *Richmond* [1973] have previously included the line-tying effect in addition to the hot particle effects; however, their analysis is limited to the study of electrostatic low-frequency modes, and the shear flow effect has not been included. On the other hand, *Patel* [1978], *Migliuolo and Patel* [1981], and *Migliuolo* [1983] have studied the electromagnetic modes, including the hot particle effects from a kinetic treatment but neglecting both the line-tying and the shear flow effects.

This paper is organized in the following manner. Section 2 presents a derivation of the general hydromagnetic wave equation from the MHD theory. A local analysis of waves propagating parallel and perpendicular to the magnetic field is presented in section 3. In section 4 we investigate the wave equation and the dispersion relation of hydromagnetic waves for a nonuniform linear plasma model. The application to the plasmapause boundary of the theoretical and numerical results of the dispersion relation for a nonuniform linear model is presented in section 5. Finally, a summary and conclusions of the results obtained are presented in section 6.

2. Mathematical Formulation of the Wave Equation

To investigate the stability of hydromagnetic fluctuations, we derive the general wave equation for small-amplitude perturbations from the MHD equations in a spherical coordinate system (r, θ, ϕ) as shown in Figure 1. The basic MHD conservation equations governing the motion of the plasma, when all transport processes (viscosity, resistivity, thermal diffusivity, etc.) are neglected, are given by

$$\rho \frac{D\mathbf{V}}{Dt} = -\nabla P + \mathbf{J} \times \mathbf{B} + \rho[\mathbf{g} - \mathbf{\Omega}_e \times (\mathbf{\Omega}_e \times \mathbf{r}) - 2\mathbf{V} \times \mathbf{\Omega}_e] \quad (1)$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0 \quad (2)$$

$$\nabla \times \mathbf{E} = \frac{\partial \mathbf{B}}{\partial t} \qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} \qquad \nabla \cdot \mathbf{B} = 0 \quad (3)$$

$$\frac{D}{Dt}(P/\rho^\gamma) = 0 \qquad \gamma = 5/3 \quad (4)$$

$$\mathbf{E} = -\mathbf{V} \times \mathbf{B} \quad (5)$$

where we define $D/Dt = \partial/\partial t + \mathbf{V} \cdot \nabla$ as the convective derivative and where ρ denotes the plasma density, $\mathbf{V}$ is the convective Eulerian velocity vector, $\mathbf{B}$ is the magnetic field, $\mathbf{E}$ is the electric field, $\mathbf{J}$ is the current density, $\mathbf{g}$ is the gravitational acceleration, $\mathbf{\Omega}_e$ is the earth angular velocity, P is the thermal pressure (assumed to be a scalar in this paper), and μ_0 is the magnetic permeability.

For simplicity we assumed a plasmapause model represented by a spherical thin shell of radius $\mathbf{r}$, where the magnetofluid properties vary linearly as a function of radial distance as shown in Figure 1. Furthermore, the magnetofluid properties at both sides of the transition zone representing the plasmapause are assumed to be constant. It is also assumed that the region inside the thin shell (i.e., the plasmasphere) is corotating with angular velocity $\mathbf{\Omega}_e$ about the z axis and that the magnetic field can be represented by the dipole field $\mathbf{B}(r, \theta)$ whose axis is aligned with the rotation axis. The determination of the wave equation for the hydromagnetic fluctuations at this transition zone rests on two assumptions: (1) the wavelengths of the disturbances under consideration are larger than the local ion Larmor radius but short in comparison to the curvature of the field lines at the boundary position; and (2) the ratio of the variation thickness of the magnetofluid properties to the radial distance of the boundary is very small. The first assumption represents a "weak curvature" condition and is only valid at low latitudes where $B_\theta(r, \theta) \gg B_r(r, \theta)$ for a dipole field. The second condition is denoted as the "narrow gap" approximation [*Acheson*, 1973; *Acheson and Gibbons*, 1978; *Howard and Gupta*, 1962].

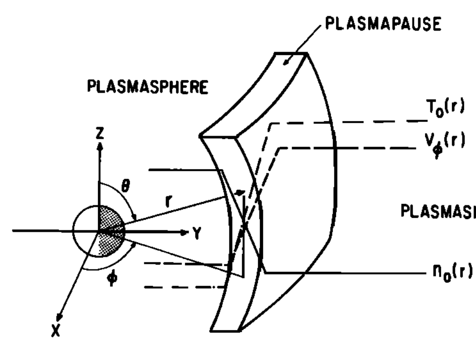

Fig. 1. Schematic of the coordinate system and the velocity profile used in the calculations. The r component is the direction of the inhomogeneity. The total velocity jump is U_0, and the scale length of the velocity shear is Δ.

The linear equilibrium magnetofluid variables at low latitudes in the equatorial plane are assumed to depend only on the radial distance, so that $\rho_0 = \rho_0(r)$ and $P_0 = P_0(r)$. The equilibrium flow velocity model $\mathbf{V}_0 = V_\phi(r)\hat{e}_\phi$ assumes that the plasma inside and outside the boundary convects in the longitudinal direction, tangential to the boundary. The magnetic field at low latitudes is given by $\mathbf{B}_0 = -B_\theta(r)\hat{e}_\theta$, and the angular velocity is chosen to be $\mathbf{\Omega}_e = \Omega_e\hat{e}_r$. The equilibrium condition for these model parameters is obtained from the radial component of the momentum equation (1) as

$$\frac{\partial}{\partial r}\left[P_0(r) + \frac{B_\theta^2(r)}{2\mu_0}\right] = -\rho_0(r)\left[g_e + \frac{C_a^2(r)}{r}\right] \tag{6}$$

where $g_e = g - [V_\phi(r) + \Omega_e r]^2/r$ is an effective gravity due to inertial forces in a rotating frame and where $C_a(r) = B_\theta(r)/(\rho_0\mu_0)^{1/2}$ is the Alfvén speed. Equation (6) shows the balance between the total ambient pressure forces (thermal plus magnetic) and the buoyancy forces. Note that the second term on the right-hand side of (6) gives an apparent gravity due to the curvature of the magnetic field lines, giving rise to an additional buoyancy force.

The linearized wave equation is obtained from (1)–(5), together with the equilibrium magnetofluid variables assuming perturbations about the equilibrium state of the form $\tilde{f}(r)$ exp $(-i\omega t + im\theta + in\phi)$. Variables with a tilde represent the fluctuating amplitudes. For the sake of simplicity the details of this derivation are presented in the appendix. However, to summarize the results, the final solution of the linearized wave equation is given by a coupled system of first-order differential equations of the form

$$\frac{\partial \tilde{X}(r)}{\partial r} = \mathsf{A}(r)\tilde{X}(r) \tag{7}$$

where the vector $\tilde{X}(r)$ gives the amplitude fluctuation defined by

$$\tilde{X}(r) = \begin{pmatrix} \tilde{P}_t \\ r^2\tilde{\xi}_r \end{pmatrix} \tag{8}$$

for which $\tilde{P}_t$ and $\tilde{\xi}_r$ are defined as follows: $\tilde{P}_t(r) = \tilde{P}(r) + B_\theta(r)\tilde{B}_\theta(r)/\mu_0$ and $\tilde{\xi}_r = i\tilde{V}_r(r)/\Omega(r)$ ($\Omega(r) = \omega - k_\phi V_\phi(r)$ is the Doppler-shifted frequency), representing the total pressure fluctuation and the radial displacement of a fluid element, respectively. In (7) we have also defined the two-by-two coupling matrix A, whose elements are given by

$$a_{11}(r) = -\left[\frac{\Omega^2(g_e + 2C_a^2/r) - 2k_\theta^2C_a^2C_s^2/r}{(C_a^2 + C_s^2)\Omega_m^2} - \frac{2\Omega k_\phi(V_\phi + \Omega_e r)}{r\Omega_a^2}\right] \tag{9a}$$

$$a_{12}(r) = \frac{\rho_0}{r^2}\left\{\Omega_a^2 - \Omega_r^2 + g_e\frac{\partial \ln \rho_0}{\partial r} - \left(g_e + \frac{2C_a^2}{r}\frac{\Omega^2}{\Omega_a^2}\right) \cdot \gamma^{-1}h_1(r)\frac{\partial \ln P_0}{\partial r} - \left(g_e - \frac{2C_s^2}{r}\right)h_2(r)\left[\frac{\partial \ln B_\theta}{\partial r} - \frac{h_3(r)}{h_2(r)r}\right]\right\} \tag{9b}$$

$$a_{21}(r) = -\frac{r^2}{\rho_0}\left[\frac{\Omega^4}{(C_a^2 + C_s^2)\Omega_m^2\Omega_a^2} - \frac{k_\theta^2 + k_\phi^2}{\Omega_a^2}\right] \tag{9c}$$

$$a_{22}(r) = -a_{11}(r) \tag{9d}$$

To simplify the tedious algebraic manipulation, we have also defined the following quantities:

$$h_1(r) = \frac{\Omega_a^2}{\Omega_m^2}(1 + C_a^2/C_s^2)^{-1}$$

$$h_2(r) = \frac{C_a^2}{C_s^2}\left[1 - \frac{C_a^2}{C_s^2}\frac{\Omega^2}{\Omega_m^2}(1 + C_a^2/C_s^2)^{-1}\right]$$

$$h_3(r) = \frac{C_a^2}{C_s^2}\left[1 - \left(\frac{\Omega^2 - 2k_\theta^2C_s^2}{\Omega_m^2}\right)\frac{C_a^2}{C_s^2}(1 + C_a^2/C_s^2)^{-1}\right]$$

$$\Omega_a^2 = \Omega^2 - k_\theta^2C_a^2 \qquad \Omega_m^2 = \Omega^2 - \frac{k_\theta^2C_a^2C_s^2}{(C_a^2 + C_s^2)}$$

$$\Omega_r^2 = \frac{4\Omega_e^2\Omega^2}{\Omega_a^2} + \frac{4\Omega_e\Omega^2V_\phi}{r\Omega_m^2}$$

The quantities $k_\theta = m/r$ and $k_\phi = n/r$ represent the latitudinal (parallel) and longitudinal (perpendicular) wave vector components, respectively. Similarly, we define $C_s(r) = [\gamma P_0(r)/\rho_0(r)]^{1/2}$ as the sound speed.

This final set of first-order coupled differential equations for the amplitude fluctuations forms the basic system of equations that will be our starting point for investigating the stability of hydromagnetic fluctuations.

3. Local Analysis of Wave Modes in a Quasi-Uniform Medium

In this section we shall present an analysis of those modes that propagate strictly parallel (i.e., $k_\| = k_\theta$, $k_\phi = 0$) and perpendicular (i.e., $k_\perp = k_\phi$, $k_\theta = 0$) to the ambient magnetic field using the wave equation (7) derived in the previous section. In order to make the basic physical phenomena more transparent a series of approximations are assumed which are consistent with the relevant plasma environment at the plasmapause.

Parallel Modes ($k_\phi = 0$)

The relevant field-aligned modes will have phase velocities in the range of $C_s < \omega/k_\| < C_a$, since this is a good approximation for the plasmapause boundary. In this situation the matrix elements (9) of the wave equation (7) reduce to

$$a_{11}(r) = -\left[\frac{\omega^2(g_e + 2C_a^2/r) - 2k_\|^2C_a^2C_s^2/r}{(C_a^2 + C_s^2)\omega^2}\right] \tag{10a}$$

$$a_{12}(r) = \frac{\rho_0}{r^2}\left[|\omega^2 - k_\|^2C_a^2| - \Omega_g^2(r)\right] \tag{10b}$$

$$a_{21}(r) = -\frac{r^2}{\rho_0|\omega^2 - k_\|^2C_a^2|}\left(\frac{\omega^2}{C_a^2 + C_s^2} - k_\|^2\right) \tag{10c}$$

$$a_{22}(r) = -a_{11}(r) \tag{10d}$$

where $\Omega_g^2(r)$ is defined by

$$\Omega_g^2(r) = -g_e\frac{\partial \ln \rho_0}{\partial r} + \frac{(g_e + 2C_a^2/r)}{\gamma(1 + C_a^2/C_s^2)}\frac{\partial \ln P_0}{\partial r} + \frac{C_a^2}{C_s^2}\frac{(g_e - 2C_s^2/r)}{(1 + C_a^2/C_s^2)}\left(\frac{\partial \ln B_\theta}{\partial r} - \frac{1}{r}\right) \tag{11}$$

which represents a measure of the oscillation rate due to the action of restoring forces that act upon a parcel of plasma which has been displaced radially by perturbations to a stable configuration. The quantity $\Omega_g(r)$, known in plasma physics as the interchange or Rayleigh-Taylor frequency, is also called the magnetic Brunt-Väisälä (BV) frequency in analogy with its hydrodynamic counterpart.

For the sake of simplicity in this analysis we modify the wave equation (7) to the form

$$\frac{\partial \tilde{Y}(r)}{\partial r} = \begin{pmatrix} 0 & b_{12}(r) \\ b_{21}(r) & 0 \end{pmatrix} \tilde{Y}(r) \qquad (12)$$

for which the matrix elements b_{ij} are given in terms of (10) by

$$b_{12}(r) = a_{12}(r) \exp\left\{-\int^r [a_{11}(r') - a_{22}(r')]\, dr'\right\}$$

$$b_{21}(r) = a_{21}(r) \exp\left\{\int^r [a_{11}(r') - a_{22}(r')]\, dr'\right\}$$

and $\tilde{Y}(r)$ is now defined by

$$\tilde{Y}(r) = \begin{pmatrix} \tilde{P}_t \exp\left[-\int^r a_{11}(r')\, dr'\right] \\ r^2 \xi_r \exp\left[-\int^r a_{22}(r')\, dr'\right] \end{pmatrix}$$

This form of the wave equation (12) is very useful in determining the propagation properties of field-aligned hydromagnetic modes since the determinant of the matrix gives their dispersion relation. Therefore, if the scale of variation of the medium properties is small in comparison to the radial wavelength, we can assume $\tilde{Y}(r)$ of the form exp $(ik_r r)$ to determine the dispersion relation

$$k_r^2 = k_{\parallel}^2\left[\frac{\Omega_g^2(r)}{|\omega^2 - k_{\parallel}^2 C_a^2|} - 1\right] + \frac{\omega^2[|\omega^2 - k_{\parallel}^2 C_a^2| - \Omega_g^2(r)]}{(C_a^2 + C_s^2)|\omega^2 - k_{\parallel}^2 C_a^2|} \qquad (13)$$

To understand the nature of the radial wave number k_r, two different situations are considered. In the first case we consider high-frequency modes such that $|\omega^2 - k_{\parallel}^2 C_a^2| \gg |\Omega_g^2(r)|$. In this case the dispersion relation (13) reduces to

$$\omega^2 \simeq (k_r^2 + k_{\parallel}^2)(C_a^2 + C_s^2) \qquad (14)$$

Modes represented by this dispersion relation correspond to fast magnetoacoustic waves or compressional Alfvén modes (since $k_r \gg k_{\parallel}$) which in this frequency regime are not affected by variations in the medium properties. The second situation corresponds to low-frequency eigenmodes such that $|\omega^2 - k_{\parallel}^2 C_a^2| \ll |\Omega_g^2(r)|$. In this case the dispersion relation (13) reduces to

$$\omega^2 \simeq k_{\parallel}^2 C_a^2(r) + \Omega_g^2(r) \qquad (15)$$

assuming that $k_r \gg k_{\parallel}$. Waves represented by (15) are shear Alfvén waves. An inspection of (15) reveals that these modes can become unstable if

$$\Omega_g^2(r) < -k_{\parallel}^2 C_a^2(r) \qquad (16)$$

This result represents the instability criterion for "ballooning" modes, which are fundamentally an interchange mode that is driven by the presence of a pressure gradient in an unfavorable magnetic field curvature [*Gold*, 1959; *Sonnerup and Laird*, 1963; *Coppi et al.*, 1979]. Note that for a dipole field which varies as r^{-3}, this condition yields

$$-r\frac{\partial \ln P_0}{\partial r} > 4\gamma + \gamma k_{\parallel}^2 r^2 (1 + C_a^2/C_s^2) \qquad (17)$$

assuming that $g_e \ll 2C_a^2/r$ and $2C_s^2/r$. Moreover, note that the presence of finite parallel wavelength produces stabilizing effects on these modes, in agreement with previous results [*Coppi et al.*, 1979]. Since the most unstable modes are those for which the magnetic field remains unchanged [*Gold*, 1959; *Sonnerup and Laird*, 1963; *Coppi et al.*, 1979], then $k_{\parallel} = 0$, and the condition (17) can be written as

$$-\left(\frac{\partial \ln n_0}{\partial \ln r} + \frac{\partial \ln T_0}{\partial \ln r}\right) > 4\gamma \qquad (18)$$

assuming that $P_0 = n_0 K_B T_0$ and $\rho_0 = M_i n_0$. Equation (18) is consistent with that obtained by *Gold* [1959] and *Sonnerup and Laird* [1963] by means of the energy principle. This condition means that if the pressure decreases much faster than $r^{-4\gamma}$, then the system will spontaneously become convectively unstable to interchange.

Perpendicular Modes ($k_\theta = 0$)

We now proceed to study the behavior of those modes that propagate perpendicular to the magnetic field (i.e., $k_\perp = k_\phi$). For simplicity, let us neglect those terms associated with rotational effects, since for the earth's plasmasphere they are very small in comparison to the thermal and magnetic effects. Therefore, subject to these conditions, the matrix elements (9) of the wave equation (7) reduce to

$$a_{11}(r) = -\left[\frac{(g_e + 2C_a^2/r)}{(C_a^2 + C_s^2)} - \frac{2k_\perp}{r\Omega}(V_\phi + \Omega_e r)\right] \qquad (19a)$$

$$a_{12}(r) = \frac{\rho_0}{r^2}(\Omega^2 - \Omega_g^2) \qquad (19b)$$

$$a_{21}(r) = -\frac{r^2}{\rho_0}\left[\frac{1}{(C_a^2 + C_s^2)} - \frac{k_\perp^2}{\Omega^2}\right] \qquad (19c)$$

$$a_{22}(r) = -a_{11}(r) \qquad (19d)$$

Similarly to the previous section, the dispersion relation of these modes can be determined from the determinant of the matrix wave equation (7) using (19) and assuming fluctuations of the form exp $(ik_r r)$. This is valid if the medium properties vary smoothly in comparison to the radial wavelength. Therefore the dispersion relation results in

$$k_r^2(r) = k_\perp^2\left[\frac{\Omega_g^2(r)}{\Omega^2(r)} - 1\right] + \frac{[\Omega^2(r) - \Omega_g^2(r)]}{(C_a^2 + C_s^2)} \qquad (20)$$

Similarly to the parallel mode case, two different limits are considered. In the first case we considered high-frequency modes such that $|\Omega^2| \gg |\Omega_g^2|$, resulting in the dispersion relation

$$\omega = k_\perp V_\phi(r) \pm [(k_r^2 + k_\perp^2)(C_a^2 + C_s^2)]^{1/2} \qquad (21)$$

Eigenmodes represented by this relation correspond to fast magnetoacoustic waves which are nondispersive and propagate only if the condition $|\Omega/k_\perp| > (C_a^2 + C_s^2)^{1/2}$ is satisfied.

The second limit corresponds to low-frequency modes that satisfy the condition $|\Omega^2| \ll |\Omega_g^2(r)|$. The dispersion relation (20) reduces in this case to

$$\omega = k_\perp V_\phi(r) \pm \frac{k_\perp \Omega_g(r)}{(k_r^2 + k_\perp^2)^{1/2}} \qquad (22)$$

Equation (22) gives the propagation properties for Alfvén drift waves which propagate perpendicular to both the magnetic field and the gradients. These waves are dispersive and can only propagate if the condition $|\Omega(r)/k_\perp| < (C_a^2 + C_s^2)^{1/2}$ is satisfied. Note also that these modes can be unstable if $\Omega_g(r)$ becomes imaginary because of the gradient effects. In this case the growth rate of these modes is given by Im $\omega = k_\perp$ Im $\Omega_g/(k_r^2 + k_\perp^2)^{1/2}$.

It is interesting to note that these two eigenmodes, i.e., fast magnetoacoustic and Alfvén drift waves, are the hydromagnetic analogue to sound and gravity waves in hydrodynamic flows, respectively [*Miles*, 1961, 1963; *Howard*, 1961, 1963; *Booker and Bretherton*, 1967]. Furthermore, it can be shown that our problem reduces smoothly to the hydrodynamic case as the magnetic field goes to zero. For this reason we have kept the real gravity term g_e in the equations even when this term is much smaller than the apparent gravity due to the curvature of the field lines.

Although the local analysis of the dispersion relation (22) shows some of the important properties of the perpendicular modes in a quasi-uniform medium, the effect of the shear flow $V_\phi(r)$ on the stability of these modes is not quite clear from these equations. In the next section we investigate the effect of the velocity shear on the stability of these modes. We will demonstrate that even when the magnetic BV frequency $\Omega_g(r)$ is real, which represents a convectively stable system, Alfvén drift waves can become unstable if the shear flow is large enough.

4. Wave Equation and Dispersion Relation for a Nonuniform Plasma Model

In the previous sections we presented a local analysis of the nature and behavior of parallel and perpendicular modes for a quasi-uniform medium. We have shown that the stability of parallel modes or shear Alfvén waves is controlled by the nature of the magnetic BV frequency $\Omega_g(r)$ and that the most unstable modes are those for which $k_\parallel = 0$. Furthermore, we showed that in the case of perpendicular modes, low-frequency Alfvén drift waves such that $|\Omega^2| \ll |\Omega_g^2|$ can propagate if the condition $|\Omega/k_\perp| < (C_s^2 + C_a^2)^{1/2}$ is satisfied. In this section we investigate the effects of the shear flow on the stability of quasi-perpendicular modes in a nonuniform medium, since for parallel modes the flow effects vanish. Similarly to previous sections, we will consider that the rotational effects are negligible and that the sound velocity is smaller than the Alfvén velocity (which is a good approximation for the plasmapause). We further assume that these quasi-perpendicular modes are characterized by $k_\perp > k_\parallel$, since this condition corresponds to the Alfvén drift mode. Subject to these conditions, the matrix elements (9) of the general wave equation (7) reduce to

$$a_{11}(r) = -\left[\frac{(g_e + 2C_a^2/r)}{C_s^2 + C_a^2} - \frac{2k_\perp \Omega V_\phi}{r(\Omega^2 - k_\parallel^2 C_a^2)} - \frac{2k_\parallel^2 C_s^2 C_a^2}{r\Omega^2(C_s^2 + C_a^2)}\right] \qquad (23a)$$

$$a_{12}(r) = \frac{\rho_0}{r^2}\left[\Omega^2 - k_\parallel^2 C_a^2 - \Omega_g^2(r)\right] \qquad (23b)$$

$$a_{21}(r) = \frac{r^2(k_\parallel^2 + k_\perp^2)}{\rho_0(\Omega^2 - k_\parallel^2 C_a^2)} \qquad (23c)$$

$$a_{22}(r) = -a_{11}(r) \qquad (23d)$$

where $\Omega_g^2(r)$ is defined in (11). To simplify the analysis, we transformed the matrix wave equation (7) into a second-order differential equation. The final expression gives

$$\tilde{W}''(r) + p(r)\tilde{W}'(r) + q(r)\tilde{W}(r) = 0 \qquad (24)$$

where the primes indicate derivatives with respect to r and where we define

$$\tilde{W}(r) = r^2\xi_r \exp\left[-\int a_{11}(r')\,dr'\right]$$

$$p(r) = \left(-\frac{1}{r_\rho} + 2\frac{\partial \ln \Omega(r)}{\partial r}\right) \qquad r_\rho = \left(-\frac{\partial \ln \rho_0}{\partial r}\right)^{-1}$$

$$q(r) = \frac{(k_\perp^2 + k_\parallel^2)[\Omega_g^2(r) + k_\parallel^2 C_a^2]}{\Omega^2} - (k_\perp^2 + k_\parallel^2)$$

assuming that $|k_\perp(r - r_c)\partial V_\phi/\partial r| \gg |k_\parallel C_a|$ and $|\partial \ln \rho_0/\partial r| \gg |\partial \ln B_0/\partial r|$, which are valid approximations at the plasmapause for a dipole magnetic field. Equation (24) is almost identical to the hydrodynamic equation for stability of stratified flows subject to a uniform vertical downward gravitational field [*Miles*, 1961, 1963; *Howard*, 1961, 1963]. This equation can be studied by techniques similar to those used in the problem of hydrodynamical stratified shear flow to investigate unstable solutions [*Howard*, 1961, 1963]. It is possible to find unstable solutions even if the system is convectively stable, for which $\Omega_g^2(r) > 0$. Suppose (24) and the boundary conditions have a nontrivial solution $\tilde{W}(r)$ with Im $(C) > 0$ (for instability), where C is the phase velocity given by $C = \omega/(k_\perp^2 + k_\parallel^2)^{1/2}$. Then $[V_\phi(r) - C]$ does not vanish across the shear zone between r_- and r_+, and we can form a square root $[V_\phi(r) - C]^{1/2}$ which is as smoothly varying as $V_\phi(r)$. We assume $V_\phi(r)$ to be continuous and differentiable. Let

$$\tilde{G}(r) = [V_\phi(r) - C]^{1/2} \exp\,[r/(2r_\rho)]\tilde{W}$$

and write (24) in terms of $\tilde{G}(r)$. The result gives

$$\frac{d}{dr}\left[(V_\phi - C)\frac{d\tilde{G}}{dr}\right] + \tilde{G} \cdot \left\{\frac{(\Omega_g^2 + k_\parallel^2 C_a^2)(1 + k_\parallel^2/k_\perp^2) - \frac{1}{4}(\partial V_\phi/\partial r)^2}{V_\phi - C} - \frac{(V_\phi - C)[4(k_\parallel^2 + k_\perp^2) + r_\rho^{-2}]}{4} - \left(\frac{1}{2}\frac{\partial^2 V_\phi}{\partial r^2} - \frac{1}{r_\rho}\frac{\partial V_\phi}{\partial r}\right)\right\} = 0 \qquad (25)$$

where we have neglected terms related to rotational forces and assumed that $(2r_\rho^{-2})^{1/2} \gg 2/r$. Multiplying (25) by $\tilde{G}^*$ (where the asterisk represents the complex conjugate) and assuming that $\tilde{G}(r_-) = \tilde{G}(r_+) = 0$, we get

$$\int_{r_-}^{r_+} (V_\phi - C)\left[\left|\frac{d\tilde{G}}{dr}\right|^2 + (k_\perp^2 + k_\parallel^2 + r_\rho^{-2}/4)|\tilde{G}|^2\right] dr + \int_{r_-}^{r_+}\left(\frac{1}{2}\frac{\partial^2 V_\phi}{\partial r^2} - \frac{1}{r_\rho}\frac{\partial V_\phi}{\partial r}\right)|\tilde{G}|^2\,dr + \int_{r_-}^{r_+}(V_\phi - C^*)\left[\frac{1}{4}\left(\frac{\partial V_\phi}{\partial r}\right)^2 - (\Omega_g^2 + k_\parallel^2 C_a^2)(1 + k_\parallel^2/k_\perp^2)\right]\left|\frac{\tilde{G}}{V_\phi - C}\right|^2 dr = 0$$

The imaginary part of this equation, if $C_i = \text{Im}\,(C)$, gives

$$\int_{r_-}^{r_+}\left[\left|\frac{d\tilde{G}}{dr}\right|^2 + (k_\perp^2 + k_\parallel^2 + r_\rho^{-2}/4)|\tilde{G}|^2\right] dr + \int_{r_-}^{r_+}\left[(\Omega_g^2 + k_\parallel^2 C_a^2)(1 + k_\parallel^2/k_\perp^2) - \frac{1}{4}\left(\frac{\partial V_\phi}{\partial r}\right)^2\right]\left|\frac{\tilde{G}}{V_\phi - C}\right|^2 dr = 0$$

which is impossible to satisfy if $[\Omega_g^2(r) + k_{\parallel}^2 C_a^2](1 + k_{\parallel}^2/k_{\perp}^2)$ is everywhere greater than or equal to $(\partial V_\phi/\partial r)^2/4$. Thus a sufficient condition for instability is that

$$Ri = \frac{[\Omega_g^2(r) + k_{\parallel}^2 C_a^2](1 + k_{\parallel}^2/k_{\perp}^2)}{(\partial V_\phi/\partial r)^2} < \frac{1}{4} \tag{26}$$

The dimensionless parameter Ri is called the local magnetic Richardson number in analogy with hydrodynamic flows. Furthermore, we find that unstable modes (i.e., Im $(C) > 0$) satisfy Howard's semicircle theorem [*Howard*, 1961, 1963]. This theorem states that for complex wave speed C such that Im $(C) > 0$ any unstable mode must lie inside a semicircle in the upper half C plane which has a range of $V_\phi(r)$ in diameter if $\Omega_g^2(r) > 0$ [*Howard and Gupta*, 1962; *Howard*, 1961, 1963].

Two important conclusions are obtained by inspecting the magnetic Richardson condition (26). First, note that the interchange instability (i.e., $\Omega_g^2 < -k_{\parallel}^2 C_a^2$) cannot occur before the shear instability since the magnetic Richardson number will become smaller than 0.25 even for small shears. Second, note that the direction of the velocity flow $V_\phi(r)$ perpendicular to the magnetic field is not important, since the gradient term appears to the second power in (26). Equation (26) also shows the coupling between the parallel shear Alfvén mode and the perpendicular Alfvén drift mode as well as the stabilizing influence of the finite parallel wavelength in agreement with previous results.

Let us now determine the solutions of the wave equation (24) inside and outside the shear flow transition zone. As we previously mentioned, the plasmasphere model is represented by a thin spherical shell at a distance r from the earth's center as shown in Figure 1. We assumed the narrow gap approximation and considered the scale length of variations of the medium properties to be constant across the transition zone. For simplicity we also assumed that the medium properties inside and outside the layer are constant. Subject to these conditions, the solution of the wave equation (24) outside the transition region gives

$$\tilde{W}(r) = B_1 e^{kr} + B_2 e^{-kr} \tag{27}$$

where $k^2 = k_{\perp}^2 + k_{\parallel}^2$ is the total horizontal wave vector and B_1 and B_2 are constants of integration to be determined from the boundary conditions.

To determine the solution of the wave equation in the transition layer, we transform (24) by expanding the Doppler-shifted frequency $\Omega(r)$ in a Taylor series around some point r' inside the region. This expansion gives

$$\frac{1}{\Omega(r)} \simeq \frac{(-k_{\perp}\,\partial V_\phi/\partial r)^{-1}|_{r=r_c}}{r - r_c} \tag{28}$$

where r_c is defined by

$$r_c = r' + \frac{\text{Re }\Omega(r') + i\omega_i}{k_{\perp}\,\partial V_\phi/\partial r|_{r=r'}} \tag{29}$$

Finally, we make the substitutions

$$\tilde{W}(r) = \frac{\tilde{\psi}(r)\exp(r/2r_\rho)}{r - r_c} \qquad \hat{k} = k\Delta \qquad \mu = (\tfrac{1}{4} - Ri)^{1/2} \tag{30}$$

$$\zeta = \frac{(r - r_c)}{\Delta}(4\hat{k}^2 + \Delta^2/r_\rho^2)^{1/2} \qquad \alpha = \frac{\Delta/r_\rho}{(4\hat{k}^2 + \Delta^2/r_\rho^2)^{1/2}} \tag{31}$$

where Δ is the thickness of the shear flow profile. Therefore the wave equation (24) reduces to

$$\frac{d^2\tilde{\psi}(\zeta)}{d\zeta^2} + \left(-\frac{1}{4} + \frac{\alpha}{\zeta} + \frac{\frac{1}{4} - \mu^2}{\zeta^2}\right)\tilde{\psi}(\zeta) = 0 \tag{32}$$

which is the well-known Whittaker's differential equation. The solutions of this equation can be written in terms of the confluent hypergeometric functions as follows:

$$\tilde{\psi}(\zeta) = C_1 M_{\alpha,\mu}(\zeta) + C_2 M_{\alpha,-\mu}(\zeta) \tag{33}$$

where C_1 and C_2 are integration constants and $M_{\alpha,\pm\mu}(\zeta)$ is given by

$$M_{\alpha,\pm\mu}(\zeta) = \zeta^{(1/2\pm\mu)} e^{-\zeta/2}\ {}_1F_1(\tfrac{1}{2} - \alpha \pm \mu, 1 \pm 2\mu; \zeta) \tag{34}$$

where the function ${}_1F_1(a, b; \zeta)$ is the confluent hypergeometric function.

Note that (33) has a branch point at $\zeta = 0$ (i.e., $r = r_c$). This occurs when the real part of the Doppler-shifted frequency vanishes inside the transition zone. Therefore $\zeta = 0$ corresponds to a critical level at which the phase velocity ω/k couples to the flow velocity $V_\phi(r_c)$. Physically, these critical levels represent sites where strong coupling between Alfvén drift waves and the background flowing plasma occurs. These singularities have come to be known as critical levels, since as the mathematical concept suggests, the wave behavior can be rather dramatic [*Miles*, 1961, 1963; *Howard*, 1961, 1963; *Booker and Bretherton*, 1967]. Since we are looking for unstable modes such that $C_i = \text{Im }(C) \gg 0$, then we require the eigenfunctions $\tilde{\psi}(\zeta)$ to be continuous and differentiable across the branch point $\zeta = 0$. Thus, if there exists any unstable solution in the upper half C plane, we shall take the path of integration below the branch point as we cross it, restricting the argument of ζ according to [*Miles*, 1961, 1963; *Booker and Bretherton*, 1967]

$$-\pi < \arg(\zeta) < 0 \qquad \text{Im }(C) > 0 \tag{35}$$

With this condition the analytic continuation of (33) around the branch point $\zeta = 0$ can be determined according to

$$M_{\alpha,\pm\mu}(\zeta) = e^{-i\pi(1/2\pm\mu)} M_{-\alpha,\pm\mu}(-\zeta) \tag{36}$$

Once the solution of the wave equation inside and outside the transition zone is known, the dispersion relation for a nonuniform medium can be determined using physically reasonable boundary conditions. The boundary conditions to be met at the lower and upper boundaries of the transition layer are the continuity of the wave impedance defined by $\tilde{Z} = \tilde{P}_t/(r^2\tilde{\xi}_r)$; however, this condition is equivalent to the continuity of the logarithmic derivative of the function $\tilde{W}(r)$. Additional conditions are applied to (27) in the region above and below the transition zone. These conditions assume that the wave is evanescent in these regions, which is justified by the fact that the Doppler-shifted frequency is much greater than the magnetic BV frequency. Therefore by matching the logarithmic derivatives of the wave solutions at the upper (r_+) and lower (r_-) boundaries of the transition layer, we get the dispersion relation

$$\begin{aligned} D(\hat{c}_r, \hat{c}_i, \hat{k}, Ri) = {} & e^{i\pi\mu}(\zeta_+/\zeta_-)^\mu f(\zeta_+, \hat{p}_+, \alpha, \mu) f(\zeta_-, \hat{p}_-, -\alpha, -\mu) \\ & - e^{-i\pi\mu}(\zeta_+/\zeta_-)^{-\mu} f(\zeta_+, \hat{p}_+, \alpha, -\mu) f(\zeta_-, \hat{p}_-, -\alpha, \mu) = 0 \end{aligned} \tag{37}$$

where we have defined

$$\zeta_+ = \frac{\Delta}{\alpha r_\rho}(1 - \hat{c}) \qquad \zeta_- = \frac{\Delta}{\alpha r_\rho}\hat{c} \qquad \hat{p}_\pm = \frac{\hat{k} \pm \Delta/r_\rho}{(4\hat{k}^2 + \Delta^2/r_\rho^2)^{1/2}}$$

$$f(\zeta_\pm, \hat{p}_\pm, \pm\alpha, \pm\mu) = (\hat{p}_\pm\zeta_\pm - \tfrac{1}{2} \pm \mu)\Phi_{\pm\alpha,\pm\mu}(\zeta_\pm) + \zeta_\pm \frac{d\Phi_{\pm\alpha,\pm\mu}}{d\zeta}(\zeta)\bigg|_{\zeta=\zeta_\pm}$$

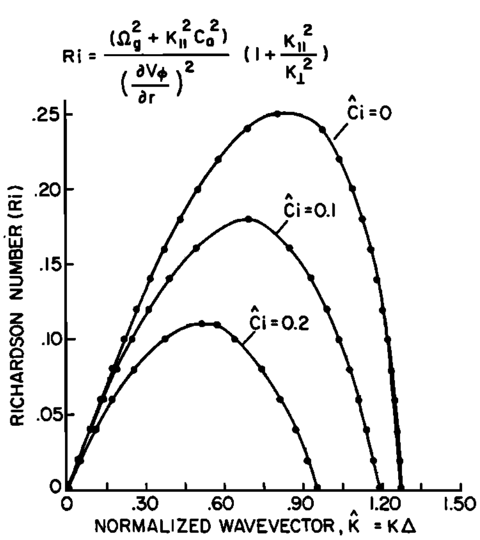

Fig. 2. Instability surfaces representing the plasma medium parameters Ri versus the possible unstable normalized wavelengths $k\Delta$ at constant normalized growth rate $\hat{c}_i$ for perpendicular modes (i.e., drift waves).

$$\Phi_{\pm\alpha,\pm\mu}(\zeta_\pm) = e^{-\zeta/2}\,{}_1F_1(\tfrac{1}{2} \mp \alpha \pm \mu, 1 \pm 2\mu, \zeta_\pm)\Big|_{\zeta=\zeta_\pm}$$

and $\hat{c}$ is the normalized complex phase velocity given by

$$\hat{c} = \frac{\hat{\omega}}{\hat{k}} = \hat{c}_r + i\hat{c}_i \qquad \hat{\omega} = (\omega_r + i\omega_i)\Delta/U_0$$

where U_0 is the maximum flow velocity.

Equation (37) together with the condition $-\pi\mu < \arg(\zeta_+/\zeta_-)^\mu < 0$ can be solved numerically using either a predictor-corrector scheme or the Muller method to determine the dispersive properties of the unstable eigenmodes. The results, in a parametric form of the dispersion relation (37), are shown in Figures 2 and 3. In the calculation of these curves we have assumed for simplicity the Boussinesq approximation by setting $\alpha = 0$ (i.e., $\Delta/r_p = 0$). The Boussinesq approximation is a convenient framework in which to develop concepts which depend essentially on the buoyancy forces and their interplay with the shear. These concepts may probably be extended into a wider context, but for the present time the approximation is adopted without comment. Figure 2 shows contour plots of the magnetic Richardson number Ri versus the normalized wave vector $\hat{k} = k\Delta$ for constant values of the normalized imaginary part of the phase velocity (i.e., the growth rate). These curves present those regions in the model parameters for which instability can be found. The boundary at which $\hat{c}_i = 0$ divides the region of instability, given by $\hat{c}_i > 0$ and $Ri < 0.25$, from the stability region given by $\hat{c}_i < 0$ and $Ri > 0.25$. Note that as the magnetic Richardson number Ri decreases, the growth rate increases for a fixed wavelength. A curious feature in the evaluation of these eigensolutions is that all the unstable modes have the same normalized real phase velocity (i.e., $\hat{c}_r = 0.5$), which corresponds to the case where the critical level or singularity is at the center of the velocity profile. This feature seems to be due to the symmetry of the model profile across the layer and to the symmetry in the boundary conditions above and below the transition zone.

Similarly, Figure 3 shows contour plots of the normalized growth rates $\hat{\omega}_i = \mathrm{Im}\,\hat{\omega}$ versus the normalized wave vector $\hat{k}$ for constant values of Ri. These curves represent the rate at which the instability grows for a particular wavelength and plasma model (i.e., magnetic Richardson number). Note again that the maximum growth rate occurs at the smallest value of Ri for a fixed wavelength. These numerical results are consistent with the calculations made by *Miura and Pritchett* [1982], although a different procedure was used. Our results show not only that modes with $k\Delta < 1.28$ are unstable but also that these modes must satisfy the condition $Ri < 0.25$.

5. Application to the Plasmapause Boundary

We now present the application of the theoretical and numerical results obtained in the previous sections to the plasmapause transition region. The parametric form of these results shown in Figures 2 and 3 allows us to study with great flexibility a large variety of physical models of the ambient plasma. As we previously stated, the plasma parameters outside and inside the plasmasphere are chosen to be constants, while across the transition region they are chosen to vary linearly with radial distance.

The plasma parameters shown in Table 1 and used in our calculation are typical of spacecraft measurements across the plasmapause. However, in situ electric field measurements and their spatial variation are difficult to obtain because of the reliability of these measurements in regions where the plasma density (temperature) is very low (high), such as outside the plasmapause. These values are necessary to determine the velocity shear near the plasmapause. Nonetheless, indirect estimates of such shears can be obtained from low-altitude ionospheric observations at the edge of the diffuse aurora (M. Kelley, private communication, 1985). Observations of large electric fields in this region have been reported by *Smiddy et al.* [1977], *Maynard* [1978], and *Rich et al.* [1980] using satellite-borne electric field detectors. *Kelly* [1986] has used these observations to evaluate the velocity shears observed in this region in order to explain the observed fluctuations at the edge of the diffuse aurora reported by *Lui et al.* [1982]. These ionospheric shear flows can be transformed to the equatorial plane as indicated by *Mozer* [1970], assuming a dipole field model and considering the magnetic field lines to be equipotentials (frozen-in lines). The shear flow mapping from the ionosphere to the magnetospheric equatorial plane has been

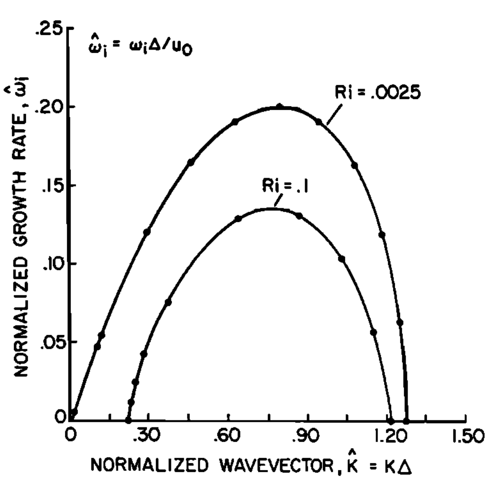

Fig. 3. Growth rate surfaces of the normalized Im ($\hat{\omega}$) versus normalized wavelengths $k\Delta$ at constant plasma medium parameters Ri for perpendicular modes.

TABLE 1. Typical Plasma Model Parameters Inside and Outside the Plasmasphere

Plasma Parameters	Inside	Outside
n, parts/cm^3	800.0	1.0
K_BT, eV	1.0	10^3

The plasmapause position is at $r = 4\ R_E$, and the scale of variation of density and temperature is 0.5 R_E. The parallel wave vector $k_\| = 3.92 \times 10^{-5}\ \text{km}^{-1}$.

recently discussed by *Kelley* [1986], who indicated that the ratio R of the ionospheric to magnetospheric shear is given by

$$R = \frac{(dV_y/dx)_i}{(dV_\phi/dr)_m} \simeq \frac{B_m}{B_i} 4L^2(L - \tfrac{3}{4})$$

where L is the McIlwain shell number and B_i and B_m are the magnetic fields at the ionosphere and magnetosphere, respectively.

An estimate of the velocity shear in the vicinity of the plasmapause during a period of enhanced activity can be obtained using the observations of *Smiddy et al.* [1977]. They have reported a peak electric field of 280 mV/m at an altitude of 1463 km for their most dramatic event and further estimated the maximum electric field shear (dE/dx) to be $\pm 7.7 \times 10^{-6}$ V/m^2. Using a value of the magnetic field at ionospheric altitude of $B_i = 0.56$ G, we estimated an ionospheric velocity shear of $(dV_y/dx)_i \simeq 0.15\ \text{s}^{-1}$ (where $V_y = E/B_i$). Since there is no evidence of parallel electric fields in this region and by using a value for the equatorial plasmapause of $L = 4$ and an ionospheric to magnetospheric field ratio of $B_i/B_m = 115$ as indicated by the observations, we estimated a velocity shear ratio of $R = 1.8$. This implies that the magnetospheric shear flow $(dV_\phi/dr)_m$ is about 0.083 s^{-1}. We now estimate the quantity $\Omega_g^2 + k_\|^2 C_a^2$ at the equatorial plasmapause. Since in this region $C_a^2 \gg C_s^2$ and rotational effects are negligible in comparison to magnetic or thermal forces, we then can assume that $g_e \ll 2C_s^2/r$ and $2C_a^2/r$. Therefore the expression for $\Omega_g^2(r)$ can be reduced to

$$\Omega_g^2(r) = \frac{v_t^2}{r}\left(\frac{\partial \ln n}{\partial r} + \frac{\partial \ln T}{\partial r} + \frac{4\gamma}{r}\right)$$

where v_t is the ion thermal velocity defined as $v_t = (2K_BT/M_i)^{1/2}$. From the plasma parameters in Table 1 we find density and temperature scale lengths of 477 km and 461 km, respectively. Similarly, we calculated the thermal velocity to be about 309.7 km/s. With these parameters we estimated Ω_g to be about 3.53×10^{-2} rad/s, corresponding to a period of 178 s. Consequently, the magnetic Richardson number Ri for this plasma model yields $Ri = 0.18$ and therefore represents an unstable situation according to the theoretical predictions. These estimates indicate that such a velocity shear is confined to a region Δ of about 928 km with a maximum flow velocity outside the layer of $U_0 = 78$ km/s, as obtained from the ionospheric to magnetospheric mapping [*Mozer*, 1970; *Kelley*, 1986]. The wave parameters associated with such magnetic Richardson number and plasma flow conditions can be determined from Figures 2 and 3. Figure 2 shows that for $Ri = 0.18$ the range of unstable normalized wavelengths $\hat{k}$ occurs between $\hat{k} = 0.43$ and $\hat{k} = 1.14$. This corresponds to a range of unstable wavelengths of 5.11×10^3 to 1.36×10^4 km. The unstable modes characterized by these wavelengths have frequencies in the range of 2.88×10^{-3} to 7.65×10^{-3} Hz, corresponding to wave periods between 131 and 348 s. The maximum growth rate ω_i associated with these eigenmodes gives 5.83×10^{-3} rad/s at a wavelength of about 8.4×10^3 km (i.e., $\hat{k}_{max} = 0.69$). This estimate of the growth rate is larger than that predicted by *Hasegawa* [1971] and consistent with that predicted by *Patel* [1978]. The frequency range of these waves is also consistent with that of Pc 4–Pc 5 micropulsations, which are low-frequency hydromagnetic waves. An interesting parameter that can be determined using these results is the critical velocity shear necessary to obtain instability. We estimated that velocity shears greater than 0.071 s^{-1} are sufficient to satisfy the condition $Ri < 0.25$; therefore the velocity shear obtained indirectly from ionospheric observations is well above such a critical value and can trigger the shear instability.

A similar analysis has been performed for another event observed by *Rich et al.* [1980] (see Figure 3 in their paper). They observe intense poleward ionospheric electric fields as high as 350 mV/m at altitudes of about 1300 km. Although they do not report their estimate of parallel current density or of electric field shear, we can nevertheless estimate a magnetospheric velocity shear by mapping the typical drift velocity caused by such an intense electric field into the equatorial plane and by assuming a similar velocity scale length Δ as in the previous example. It is found that such an electric field yields ionospheric drift velocities of about 10 km/s. Mapping such a velocity to the equatorial plane for $L = 4$, we find a velocity of about 80 km/s. Assuming then a velocity scale length of about $\Delta = 928$ km, we find a velocity shear $(\partial V_\phi/\partial r)_m = 0.09\ \text{s}^{-1}$, which, as expected, is higher than in the previous case. We estimated a magnetic Richardson number $Ri = 0.15$ for this kind of velocity shear, assuming the same plasma parameters as in Table 1. Similarly, the range of unstable wavelengths $\hat{k}$ for $Ri = 0.15$ occurs between $\hat{k} = 0.34$ and $\hat{k} = 1.18$. Wavelengths in the range of 4.94×10^3 to 1.72×10^4 km and frequencies between 2.33×10^{-3} and 8.1×10^{-3} Hz represent the unstable eigenmodes of the system. Such a frequency range corresponds to wave periods between 123 and 429 s. The maximum growth rate ω_i associated with these modes is 7.6×10^{-3} rad/s at a wavelength of about 9.26×10^3 km (i.e., $\hat{k} = 0.63$). Figures 2 and 3 show that such an increase (decrease) in the velocity shear (magnetic Richardson number) implies a larger growth rate.

It is important to mention that not all the ionospheric events shown by *Rich et al.* [1980] produce unstable situations (i.e., $Ri < 0.25$). Various events were either marginally stable ($Ri \simeq 0.25$) or fully stable ($Ri > 0.25$). Furthermore, variations in other parameters such as the temperature and density scale lengths, as well as the radial position of the transition layer, must be considered in order to investigate in more detail different plasmapause conditions. These parameters affect the magnitude and sign of $\Omega_g^2(r)$ and can stabilize or destabilize the shear instability. However, we shall not attempt in this paper to explore all the parameter ranges that can trigger the instability but instead present a physical description of this instability.

6. Summary and Conclusions

We have presented a unified linear analysis of a "shear flow–ballooning" instability to explain the excitation of low-frequency hydromagnetic waves in the inner magnetosphere using a nonuniform, compressible linear plasma model profile. The results indicate that low-frequency Alfvén drift waves can be generated by such an instability. An important aspect of this analysis is the extension into the hydromagnetic context of the Richardson instability of hydrodynamic flows by unifying the K-H instability and the convective interchange instability. Essential to this extension is the concept of magnetic buoyancy produced by an effective gravity due to the curvature of the magnetic field lines, which allows the coupling between both instabilities. As a result, we demonstrated by dynamical principles the condition of instability for ballooning modes in a dipole field, which in the past has always been derived from energy principles. An important result that arises from the concept of the magnetic Richardson number is that even in the presence of small velocity shears the convective ballooning instability cannot be excited before a shear instability, because the condition $Ri < 0.25$ is satisfied before the criterion $\Omega_g{}^2 < -k_{\parallel}{}^2 C_a{}^2$. Accordingly, compressibility effects indicated by $\Omega_g{}^2$ and magnetic buoyancy are stabilizing effects to the K-H shear flow instability.

With regard to the plasmapause case we find that if the indirect estimates of the equatorial velocity shears employed in our calculations are reasonable, then the shear flow–ballooning instability can be excited at the outer edges of this boundary. This instability may then be the source of hydromagnetic fluctuations with a predominant wave vector in the longitudinal direction (since $k_{\parallel} \ll k_{\perp}$). On this basis we find that low-frequency hydromagnetic pulsations (Pc 4–Pc 5) with typical wave periods ranging between 123 and 428 s and wavelengths in the range of 5×10^3 to 17.2×10^3 km with typical growth rate of about 7×10^{-3} rad/s can be excited at the plasmapause boundary. From the typical plasma parameters observed at the plasmapause we infer, unlike other authors, that it is very unlikely that the purely thermally or centrifugally driven interchange can operate in its outer edges to excite hydromagnetic fluctuations.

There still remain various aspects which deserve special consideration in the stability analysis: (1) to properly address the interchange mode, we must also include in the calculations the line-tying effects at the foot of the field lines due to a conductive layer (such as the ionosphere) that will tend to stabilize the ballooning instability and probably will further stabilize the shear instability; (2) a more refined treatment of the shear flow–ballooning instability must include kinetic effects; and (3) although we assumed isotropic pressure, it is also possible to find additional energy sources for instability due to anisotropic effects in the thermal pressure of the plasma.

APPENDIX

The derivation of the general hydromagnetic wave equation (7) is conveniently accomplished using the MHD equations (1)–(5) in section 2. We consider that an inhomogeneous plasma is held by gravitational, rotational, thermal, and magnetic forces in a spherical coordinate system centered at the earth. It is assumed that the plasma corotates with angular velocity Ω_e about the z axis in the presence of a dipole field whose axis is aligned to the rotation axis. The ambient equilibrium variables introduced are given in section 2 and shown in Figure 1. We consider solutions near the equatorial plane such that $B_\theta(r, \theta) \gg B_r(r, \theta)$, so that the equilibrium condition becomes

$$\frac{\partial}{\partial r}\left[P_0(r) + \frac{B_\theta{}^2(r)}{2\mu_0}\right] = -\rho_0(r)\left[g_e + \frac{C_a{}^2(r)}{r}\right] \tag{A1}$$

where g_e is an effective gravity and C_a is the Alfvén velocity defined in section 2. Fluctuations of the equilibrium state are introduced by using perturbations of the form $\tilde{f}(r)\exp(-i\omega t + im\theta + in\phi)$. Variables with a tilde represent the fluctuation amplitudes. Substituting these perturbations into the MHD equations (1)–(5) and separating them into components results in the linearized system given by

$$-i\Omega\rho_0\tilde{V}_r - \frac{2}{r}\rho_0\tilde{V}_\phi(V_\phi + \Omega_e r) = -\frac{\partial\tilde{P}_t}{\partial r} - g_e\tilde{\rho} + \frac{1}{\mu_0}\left(ik_r B_\theta\tilde{B}_r - \frac{2}{r}B_\theta\tilde{B}_\theta\right) \tag{A2}$$

$$-i\Omega\rho_0\tilde{V}_\theta = -ik_\theta\tilde{P}_t + \frac{1}{\mu_0}\left(ik_\theta B_\theta\tilde{B}_\theta + \frac{B_\theta\tilde{B}_\theta}{r} + \tilde{B}_r\frac{\partial B_\theta}{\partial r}\right) \tag{A3}$$

$$-i\Omega\rho_0\tilde{V}_\phi + 2\rho_0\Omega_e\tilde{V}_r + \frac{\rho_0\tilde{V}_r}{r}\frac{\partial}{\partial r}(rV_\phi) + \rho_0\tilde{V}_r\frac{\partial V_\phi}{\partial r} = -ik_\phi\tilde{P}_t + \frac{1}{\mu_0}ik_\theta B_\theta\tilde{B}_\phi \tag{A4}$$

$$-i\Omega\tilde{B}_r = ik_\theta B_\theta\tilde{V}_r \tag{A5}$$

$$-i\Omega\tilde{B}_\theta = ik_\theta B_\theta\tilde{V}_\theta + \frac{B_\theta\tilde{V}_r}{r} - B_\theta\left[r^{-2}\frac{\partial}{\partial r}(r^2\tilde{V}_r) + ik_\theta\tilde{V}_\theta + ik_\phi\tilde{V}_\phi\right] - \tilde{V}_r\frac{\partial B_\theta}{\partial r} \tag{A6}$$

$$-i\Omega\tilde{B}_\phi + \frac{V_\phi\tilde{B}_r}{r} = ik_\theta B_\theta\tilde{V}_\phi + \tilde{B}_r\frac{\partial V_\phi}{\partial r} \tag{A7}$$

$$-i\Omega\tilde{\rho} = -\rho_0\left[r^{-2}\frac{\partial}{\partial r}(r^2\tilde{V}_r) + ik_\theta\tilde{V}_\theta + ik_\phi\tilde{V}_\phi\right] - \tilde{V}_r\frac{\partial\rho_0}{\partial r} \tag{A8}$$

$$-i\Omega\tilde{P} + \tilde{V}_r\frac{\partial P_0}{\partial r} = C_s{}^2\left(-i\Omega\tilde{\rho} + \tilde{V}_r\frac{\partial\rho_0}{\partial r}\right) \tag{A9}$$

where $\Omega(r) = \omega - k_\phi V_\phi(r)$ is the Doppler-shifted frequency, $k_\theta = m/r$ and $k_\phi = n/r$ are the latitudinal and longitudinal wave vectors, respectively, $C_s(r)$ is the sound speed ($C_s(r) = [\gamma P_0(r)/\rho_0(r)]^{1/2}$), and $\tilde{P}_t$ is the total pressure fluctuation ($\tilde{P}_t = \tilde{P} + B_\theta\tilde{B}_\theta/\mu_0$).

Equations (A2)–(A9) can be further simplified by combining them to give the general wave equation for the perturbed amplitudes. First, we solve for the density perturbation $\tilde{\rho}$ and the magnetic field fluctuations $\tilde{B}_r$, $\tilde{B}_\theta$, and $\tilde{B}_\phi$ from the linearized equation of state (A9) and the modified Maxwell's equations (A5)–(A7) in order to obtain the perturbed velocities $\tilde{V}_\theta$ and $\tilde{V}_\phi$ from (A3) and (A4). The resulting equations for $\tilde{V}_\theta$ and $\tilde{V}_\phi$ are eliminated by substitution into the linearized continuity equation (A8) and the radial component of the momentum equation (A2). The final result is the wave equation in the form of a coupled system of first-order differential equations for the amplitude perturbation shown as follows:

$$\frac{\partial\tilde{X}(r)}{\partial r} = \mathsf{A}(r)\tilde{X}(r) \tag{A10}$$

where the vector $\tilde{X}(r)$ is defined by

$$\tilde{X}(r) = \begin{pmatrix}\tilde{P}_t \\ r^2\tilde{\xi}_r\end{pmatrix} \tag{A11}$$

Here $\tilde{\xi}_r$ is the radial displacement given as $\tilde{\xi}_r = i\tilde{V}_r/\Omega(r)$, and $\mathsf{A}(r)$ is a two-by-two coupling matrix whose elements are shown by the matrix elements (9) in section 2 of this paper.

Acknowledgments. We thank E. C. Sittler and M. Acuña for their valuable suggestions and support. We would especially like to express our appreciation to M. C. Kelley for his interesting comments and suggestions. We are grateful to the members of the Information Analysis and Display group at Goddard Space Flight Center.

The Editor thanks the two referees for their assistance in evaluating this paper.

REFERENCES

Acheson, D. J., Hydromagnetic wavelike instabilities in a rapidly rotating stratified fluid, *J. Fluid Mech.*, *61*, 609, 1973.

Acheson, D. J., and M. P. Gibbons, Magnetic instabilities of a rotating gas, *J. Fluid Mech.*, *85*, 743, 1978.

Barfield, J. N., and P. J. Coleman, Storm-related wave phenomena observed at the synchronous equatorial orbit, *J. Geophys. Res.*, *75*, 1943, 1970.

Barfield, J. N., and R. L. McPherron, Investigation of interaction between Pc 1 and 2 and Pc 5 micropulsation at synchronous orbit during magnetic storms, *J. Geophys. Res.*, *77*, 4707, 1972.

Barfield, J. N., R. L. McPherron, P. J. Coleman, and D. J. Southwood, Storm-associated Pc 5 micropulsation events observed at the synchronous equatorial orbit, *J. Geophys. Res.*, *77*, 143, 1972.

Booker, J., and F. P. Bretherton, The critical layer for internal gravity waves in a shear flow, *J. Fluid Mech.*, *27*, 513, 1967.

Chang, D. B., L. D. Pearlstein, and M. N. Rosenbluth, On the interchange stability of the Van Allen belt, *J. Geophys. Res.*, *70*, 3085, 1965. (Correction, *J. Geophys. Res.*, *71*, 351, 1966.)

Chappell, C. R., Detached plasma regions in the magnetosphere, *J. Geophys. Res.*, *79*, 1861, 1974.

Coppi, B., J. Filreis, and F. Pegoraro, Analytical representation and physics of ballooning modes, *Ann. Phys.*, *121*, 1, 1979.

Dwarkin, M. L., A. J. Zmuda, and W. E. Radford, Hydromagnetic waves at 6.25 earth radii with periods between 3 and 240 seconds, *J. Geophys. Res.*, *76*, 3668, 1971.

Fejer, J. A., Hydromagnetic stability at a fluid velocity discontintuity between compressible fluids, *Phys. Fluids*, *7*, 499, 1964.

Gerwin, R., Stability of the interface between two fluids in relative motion, *Rev. Mod. Phys.*, *40*, 652, 1968.

Gold, T., Motions in the magnetosphere of the earth, *J. Geophys. Res.*, *64*, 1219, 1959.

Hasegawa, A., Drift-wave instability at the plasmapause, *J. Geophys. Res.*, *76*, 5361, 1971.

Hasegawa, A., and L. Chen, Theory of magnetic pulsations, *Space Sci. Rev.*, *16*, 347, 1974.

Howard, L., Note on a paper of John W. Miles, *J. Fluid Mech.*, *10*, 509, 1961.

Howard, L., Neutral curves and stability boundaries in stratified flow, *J. Fluid Mech.*, *16*, 333, 1963.

Howard, L., and A. S. Gupta, On the hydrodynamic and hydromagnetic stability of swirling flows, *J. Fluid Mech.*, *14*, 463, 1962.

Kelley, M. C., Intense sheared flow as the origin of large-scale undulations of the edge of the diffuse aurora, *J. Geophys. Res.*, in press, 1986.

Kikuchi, H., Universal instability associated with the plasmapause and its role in geomagnetic micropulsation, in *Cosmic Plasma Physics*, edited by K. Schindler, p. 45, Plenum, New York, 1971.

Kikuchi, H., Micropulsations and the plasmapause, *J. Atmos. Terr. Phys.*, *38*, 1127, 1976.

Kivelson, M. G., Instability phenomena in detached plasma regions, *J. Atmos. Terr. Phys.*, *38*, 1115, 1976.

Lanzerotti, L. J., and H. Fukunishi, Relationship of characteristics of magnetohydrodynamic waves to plasma density gradients in the vicinity of the plasmapause, *J. Geophys. Res.*, *80*, 4627, 1975.

Lanzerotti, L. J., and C. G. Maclennan, Relationships of the characteristics of magnetohydrodynamic waves to plasma density gradients near $L = 4$, *J. Atmos. Terr. Phys.*, *38*, 1093, 1976.

Lanzerotti, L. J., and D. J. Southwood, Hydromagnetic waves, in *Solar System Plasma Physics*, vol. 3, edited by L. Lanzerotti, C. F. Kennel, and E. N. Parker, p. 109, North-Holland, Amsterdam, 1979.

Lanzerotti, L. J., H. Fukunishi, and L. Chen, ULF pulsation evidence for the plasmapause, 3, Interpretation of polarization and spectral amplitude studies of Pc 3 and Pc 4 pulsation near $L = 4$, *J. Geophys. Res.*, *79*, 4648, 1974.

Lanzerotti, L. J., D. B. Mellen, and H. Fukunishi, Excitation of plasma density gradients in the magnetosphere at ultralow frequencies, *J. Geophys. Res.*, *80*, 3131, 1975.

Lemaire, J., The Roche-limit of ionospheric plasma and the formation of the plasmapause, *Planet. Space Sci.*, *22*, 757, 1974.

Lemaire, J., The mechanism of formation of the plasmapause, *Ann. Geophys.*, *31*, 175, 1975.

Lemaire, J., Steady state plasmapause positions deduced from McIlwain's electric field models, *J. Atmos. Terr. Phys.*, *38*, 1041, 1976.

Lemaire, J., and L. Kowalkowski, The role of plasma interchange motion for the formation of a plasmapause, *Planet. Space Sci.*, *29*, 469, 1981.

Liu, C. S., Low-frequency drift instabilities of the ring current belt, *J. Geophys. Res.*, *75*, 3789, 1970.

Lui, A. T. Y., C.-I. Meng, and S. Ismail, Large amplitude undualations on the equatorward boundary of the diffuse aurora, *J. Geophys. Res.*, *87*, 2385, 1982.

Maynard, N. C., On large poleward-directed electric fields at subauroral latitudes, *Geophys. Res. Lett.*, *5*, 617, 1978.

Migliuolo, S., High-β theory of low-frequency magnetic pulsations, *J. Geophys. Res.*, *88*, 2065, 1983.

Migliuolo, S., and V. L. Patel, Stability theory of driftlike electromagnetic waves in two-dimensional magnetic fields, *J. Geophys. Res.*, *86*, 5553, 1981.

Miles, J., On the stability of heterogeneous shear flow, *J. Fluid Mech.*, *10*, 496, 1961.

Miles, J., On the stability of heterogeneous shear flows, 2, *J. Fluid Mech.*, *16*, 209, 1963.

Miura, A., and P. L. Pritchett, Nonlocal stability analysis of the MHD Kelvin-Helmholtz instability in a compressible plasma, *J. Geophys. Res.*, *87*, 7431, 1982.

Mozer, F. S., Electric field mapping in the ionosphere at the equatorial plane, *Planet. Space Sci.*, *18*, 259, 1970.

Ong, R. S., and N. Roderick, On the Kelvin-Helmholtz instability of the earth's magnetopause, *Planet. Space Sci.*, *20*, 1, 1972.

Patel, V. L., Low frequency drift oscillations near the plasmapause, *Geophys. Res. Lett.*, *5*, 291, 1978.

Rich, F. J., W. J. Burke, M. C. Kelley, and M. Smiddy, Observations of field-aligned currents in association with strong convection electric fields at subauroral latitudes, *J. Geophys. Res.*, *85*, 2335, 1980.

Richmond, A. D., Self-induced motions of thermal plasma in the magnetosphere and the stability of the plasmapause, *Radio Sci.*, *8*, 1019, 1973.

Sen, A. K., Effect of compressibility on Kelvin-Helmholtz instability in a plasma, *Phys. Fluids*, *7*, 1293, 1964.

Smiddy, M., M. C. Kelley, W. Burke, F. Rich, R. Sagalyn, B. Shuman, R. Hays, and S. Lai, Intense poleward-directed electric fields near the ionospheric projection of the plasmapause, *Geophys. Res. Lett.*, *4*, 543, 1977.

Sonnerup, B. U. Ö, and M. J. Laird, On magnetospheric interchange instability, *J. Geophys. Res.*, *68*, 131, 1963.

Southwood, D. J., The hydromagnetic stability of the magnetospheric boundary, *Planet. Space Sci.*, *16*, 587, 1968.

Southwood, D. J., and W. J. Hughes, Theory of hydromagnetic waves in the magnetosphere, *Space Sci. Rev.*, *35*, 301, 1983.

T. R. Madden, Building 54, Room 620, Massachusetts Institute of Technology, Cambridge, MA 02139.

A. F. Viñas, Code 692, Laboratory for Extraterrestrial Physics, NASA Goddard Space Flight Center, Greenbelt, MD 20771.

(Received August 12, 1985;
accepted October 21, 1985.)