Magnetospheric Interchange Instability

DAVID J. SOUTHWOOD[1] AND MARGARET G. KIVELSON[2]

Institute of Geophysics and Planetary Physics, University of California, Los Angeles

The interchange instability is reviewed with particular emphasis on its relevance to planetary magnetospheric processes. Results of earlier work are reviewed and drawn together by the derivation of a general stability condition, based on a small perturbation approach, for a plasma with isotropic pressure contained in a curved magnetic field in the presence of gravity. The results are not restricted to low pressure (low β) or special field geometry, and the straight field result is recovered in the appropriate limit. We use a small perturbation approach for the instability analysis. Conditions under which an effective gravity may be introduced to simulate rotational effects are discussed. Recent criticism of the Jovian ring current impoundment hypothesis which was based on the straight field line results is shown to be ill founded.

INTRODUCTION

Since *Gold*'s [1959] original paper, interchange motions have been regarded as being of great importance in terrestrial magnetospheric physics. The term interchange describes motion in which flux tubes move without affecting the curvature at any fixed point. Few would quarrel with a first-order description of the terrestrial magnetospheric circulation system as a driven interchange motion. It has to be driven for two reasons. In the inner magnetosphere, where the magnetic field is dipolar, the net motion of the ring current particles is thought to be from flux tubes at high latitudes to low. Charged particles gain energy in the process, e.g., by conserving the first two adiabatic invariants, and that energy must ultimately be provided by the driving source of the circulation system. In addition, the ionosphere forms a collisional boundary layer at the feet of the magnetospheric flux tubes, and energy must also be expended in moving the flux tube feet through the ionosphere. Through the bulk of the earth's magnetosphere the motion is magnetohydrodynamic and proceeds without large magnetic field distortion. For the most part the magnetic forces dominate the gas pressure, the dynamic pressure of the flow, and any other forces (e.g., gravity) that act.

Gold's [1959] original hypothesis was that interchange motions would take place spontaneously; inward pressure gradients in the plasma would provide a source of energy for convective overturning motion in the earth's outer ionized environment. It is now clear that the solar wind controls the circulation and that much of the charged particle population originates from the distant parts of the system rather than the inner radiation zones. Thus Gold's hypothesis is no longer tenable.

Gold's original idea may not be applicable to the high-pressure ring current region, but the interchange instability may still occur as a transient process when the system is severely disturbed. The ring current shielding process discussed by *Vasyliunas* [1972], *Jaggi and Wolf* [1973], and *Southwood* [1977] has very similar physics to the instability. Detailed studies of magnetospheric interchange instability were done by *Sonnerup and Laird* [1963]. Since the recognition that the processes that set up the ring current are likely to be interchange stable (first tested by *Nakada et al.* [1965]), there has been particular interest in the instability's role in the formation and stability of the plasmapause [*Richmond*, 1973; *Lemaire*, 1974].

The plasma circulation processes of the Jovian magnetosphere are less well defined. Centrifugal forces are more important, sources of plasma are more complex, and far less information has been recorded by spacecraft. As there is a substantial interior equatorial source of plasma (namely, the satellite, Io), a net outward transport of plasma is expected. *Melrose* [1967] first examined interchange instability for the Jovian system. Interchange instability, perhaps rather like Gold originally envisaged, could provide an important means of redistributing plasma radially. Arguments that we shall review below show that in a dipolelike field configuration it can be energetically favorable for plasma to move outward under interchange. However, after the Voyager Jovian encounters, *Siscoe et al.* [1981] pointed out that outward diffusion of Iogenic (cold) plasma through interchange would be stopped by the steep outward gradient of the hotter (ring current) particles found at the outer edge of the Io torus. They called this effect ring current impoundment of the Io plasma.

Recently, *Cheng* [1985] has called the theory of Io torus–ring current impoundment into doubt and furthermore has cast doubt on the operation of the interchange instability as understood by many previous authors from *Chandrasekhar* [1958] onward. The purpose of this paper is not so much to refute Cheng's result (in fact, we show that his instability condition is correct in appropriate circumstances) but rather to rehabilitate the previous work.

The condition derived by Cheng was derived earlier by *Tserkovnikov* [1960] and also by *Newcomb* [1961]. Newcomb showed that the condition found by Cheng was of limited use in that there exists a class of "quasi-interchange" motions that very slightly bend the field for which the stability condition is independent of the field (i.e., whose stability is governed by the familiar Schwarzschild condition for an unmagnetized stratified atmosphere [*Schwarzschild*, 1906]). The implication of Newcomb's result is that the straight field interchange motion is singular and, in general, the quasi-interchange condition,

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[1]Also at Department of Physics, Imperial College of Science and Technology, London.

[2]Also at Department of Earth and Space Sciences, University of California, Los Angeles.

the Schwarzschild condition, is the more relevant. When the undisturbed equilibrium has field lines that are curved, the singularity is removed. In particular, when gravity is negligible, the stability condition proposed by *Gold* [1959] holds. When the field is curved, we shall see that interchange motions are possible that conserve the field energy, and in these circumstances the interchange stability condition does not depend on the field strength.

The standard treatment of the interchange instability (see, for example, *Newcomb* [1961] and *Cheng* [1985]) is based on an energy principle [*Bernstein et al.*, 1958]. The energy has to be calculated to second order, and there has been some confusion over the precise form of the flux tube energy [*Cheng*, 1985]. Our approach is exactly equivalent to the many earlier works, but we adopt a small amplitude perturbation approach which makes clear what dynamical constraints are satisfied.

INTERCHANGE MOTIONS AND EQUILIBRIUM

We shall consider small amplitude departures from an equilibrium in which a plasma embedded in a magnetic field is supported by field and gas pressure gradients against a gravitational field. In the initial equilibrium one has

$$\nabla p - \mathbf{j} \times \mathbf{B} = nm\mathbf{g}$$

or alternatively,

$$\nabla(p + B^2/2\mu_0) - \mathbf{b} \cdot \nabla \mathbf{b}B^2/\mu_0 = nm\mathbf{g}$$

or

$$\nabla P_T - B^2/\mu_0\mathbf{c} - nm\mathbf{g} = 0 \qquad (1)$$

where $\mathbf{j}$ is the electrical current density, $\mathbf{b}$ is the unit vector parallel to the field $\mathbf{B}$, $\mathbf{c}$ is the field curvature, $\mathbf{b} \cdot \nabla \mathbf{b}$, and P_T is the total (plasma plus magnetic) pressure.

We now consider displacing the plasma from equilibrium. In an interchange motion, flux tubes retain their shape; the field strength may change, but the field is not bent. Accordingly, we consider displacements that are perpendicular to the unperturbed field. Furthermore, we shall have to place constraints on the manner in which the displacement varies both along and across the background magnetic field direction.

We shall proceed by considering a perturbation velocity field, $\mathbf{u}$, and a corresponding plasma displacement, $\mathbf{u}/\sigma$ (i.e., we shall take the displacement to vary with time as $\exp(\sigma t)$). It is easier to discuss stability rather than instability directly. To demonstrate stability, we shall have to show that there are no displacements for which σ^2 is real and positive. (In dissipation-free MHD there is no possibility of σ^2 being imaginary.)

GOVERNING EQUATIONS

The first-order changes in density, $n^{(1)}$, and pressure, $p^{(1)}$, are given by

$$\sigma n^{(1)} = -n(\nabla \cdot \mathbf{u}) - \mathbf{u} \cdot \nabla n \qquad (2)$$

$$\sigma p^{(1)} = -\gamma p(\nabla \cdot \mathbf{u}) - \mathbf{u} \cdot \nabla p \qquad (3)$$

where γ is the ratio of specific heats. Equation (3) is derived from the adiabatic change of pressure in the plasma rest frame.

The field change is computed from the frozen field equation

$$\sigma \mathbf{B}^{(1)} = \nabla \times (\mathbf{u} \times \mathbf{B}) \qquad (4)$$

Taking the scalar product of $\mathbf{B}/\mu_0$ with equation (4), one finds

after a little manipulation that the field pressure change is given by

$$\sigma BB^{(1)}/\mu_0 = -(\nabla \cdot \mathbf{u})B^2/\mu_0 - \mathbf{u} \cdot \nabla B^2/2\mu_0 - \mathbf{u} \cdot \mathbf{c}B^2/\mu_0 \qquad (5)$$

It is also appropriate at this point to use equation (4) to derive constraints on the allowed variations of $\mathbf{u}$ along and across the field that are inherent in the assumption of interchange motion. We introduce temporarily a local coordinate system based on the undisturbed field. We choose a coordinate, x_1, along the ambient field, such that

$$dx_1 = ds/B \qquad (6)$$

where ds is an elemental length along the field.

Now we choose the second coordinate such that it is constant along field lines. At each point along the field, x_2 points in the direction of a potential interchange motion. After an interchange displacement the locus of all particles that started on a given field line corresponds to a field line of the original field. A third orthogonal coordinate direction, x_3, is then uniquely specified at each point of the tube. We choose the coordinates x_2, x_3 such that they are flux conserving, i.e.,

$$B \, dS = dx_2 \, dx_3 \qquad (7)$$

where dS is the elemental area perpendicular to the field at each point.

Let us suppose that the scale factors for the three coordinates are h_1, h_2, h_3, and unit vectors are $\hat{\mathbf{e}}_1$, $\hat{\mathbf{e}}_2$, $\hat{\mathbf{e}}_3$, respectively. Evidently,

$$h_1 = B \qquad h_2 h_3 = B^{-1} \qquad (8)$$

It also follows that

$$\nabla x_1 = \hat{\mathbf{e}}_1/h_1 \qquad (9)$$

and there are similar relationships for the other coordinates.

The introduction of such a coordinate system allows the vector interchange motion to be represented on the local flux tube by

$$\mathbf{u} = u_2 \hat{\mathbf{e}}_2 \qquad (10)$$

Two important properties of interchange motions can be derived using the local coordinate system, (x_1, x_2, x_3). By vector manipulation of the scalar product of ∇x_2 with equation (4) one may show that

$$\mathbf{B} \cdot \nabla(u_2/h_2) = 0 \qquad (11)$$

Thus u_2/h_2 is independent of position along a given tube. The requirement expresses the fact that the displacement is proportional at each point along the field to the spacing between the field lines in the equilibrium. A subtler relation holds concerning the divergence of the flow field and the displacement:

$$\nabla \cdot \mathbf{u} = J^{-1} \, \partial(Ju_2/h_2)/\partial x_2 \qquad (12)$$

where J is the Jacobian of the coordinate transformation, i.e., $h_1 h_2 h_3$, from the Cartesian to the field-aligned system. However, by construction, J is unity, and as u_2/h_2 is independent of position along the field, it follows also that div $\mathbf{u}$ is also independent of position along $\mathbf{B}$.

The dynamical requirement on the displacement is that the first-order momentum equation be satisfied. In an interchange motion the field is not bent; thus to first order one has

$$nm\sigma\mathbf{u} = -\nabla(p^{(1)} + BB^{(1)}/\mu_0) + 2\mathbf{c}BB^{(1)}/\mu_0 + n^{(1)}m\mathbf{g}_\perp \qquad (13)$$

QUADRATIC FORM

We next develop an expression for the total energy associated with the perturbation. The initial step is to take the scalar product of equation (13) with $\mathbf{u}$. One finds

$$nm\sigma\mathbf{u}^2 = -\mathbf{u} \cdot \nabla(p^{(1)} + BB^{(1)}/\mu_0)$$
$$+ 2\mathbf{u} \cdot \mathbf{c}BB^{(1)}/\mu_0 + n^{(1)}m\mathbf{g} \cdot \mathbf{u} \qquad (14)$$

Equation (14) expresses energy conservation point by point. The right-hand side is the rate of work done (power) by the body force within the plasma; the left-hand side is the rate of change of kinetic energy.

Next we integrate equation (14) throughout the volume in which the plasma is displaced. One has

$$\int nm\sigma\mathbf{u}^2 \, dV = -\int \{\mathbf{u} \cdot \nabla[p^{(1)} + BB^{(1)}/\mu_0]$$
$$- 2\mathbf{u} \cdot \mathbf{c}BB^{(1)}/\mu_0 - n^{(1)}m\mathbf{g} \cdot \mathbf{u}\} \, dV \qquad (15)$$

where the integrals are understood to be taken over the entire volume in which the perturbations are present.

Now we make use of the vector identities

$$\nabla \cdot [\mathbf{u}p^{(1)}] = \mathbf{u} \cdot \nabla p^{(1)} + p^{(1)}\nabla \cdot \mathbf{u} \qquad (16)$$

$$\nabla \cdot [\mathbf{u}BB^{(1)}] = \mathbf{u} \cdot \nabla[BB^{(1)}] + BB^{(1)}\nabla \cdot \mathbf{u} \qquad (17)$$

and Gauss's theorem to give

$$\int nm\sigma\mathbf{u}^2 \, dV = -\int \mathbf{u}(p^{(1)} + BB^{(1)}/\mu_0) \cdot d\mathbf{S}$$
$$+ \int [(\nabla \cdot \mathbf{u})(p^{(1)} + BB^{(1)}/\mu_0)$$
$$+ 2\mathbf{u} \cdot \mathbf{c}BB^{(1)}/\mu_0 + n^{(1)}m\mathbf{g} \cdot \mathbf{u}] \, dV \qquad (18)$$

Substituting from equations (2), (3), and (5) and the equilibrium condition, (1), gives

$$\int nm\sigma^2\mathbf{u}^2 \, dV = -\int \{(\nabla \cdot \mathbf{u})^2 \, P_\gamma$$
$$+ 2(\nabla \cdot \mathbf{u})\mathbf{u} \cdot [\nabla P_T + \mathbf{c}B^2/\mu_0]$$
$$+ 2\mathbf{u} \cdot \mathbf{c}[\mathbf{u} \cdot \nabla B^2/2\mu_0 + \mathbf{u} \cdot \mathbf{c}B^2/\mu_0]$$
$$+ (m\mathbf{g} \cdot \mathbf{u})(\mathbf{u} \cdot \nabla n)\} \, dV \qquad (19)$$

where $P_\gamma = (\gamma p + B^2/\mu_0)$ and we have removed the contribution from the surface integral by taking the boundaries to be outside the region in which the plasma displacements take place.

For stability, σ^2 must be negative for all possible displacements. Let us rearrange (19) such that the cross terms in $\nabla \cdot \mathbf{u}$ and $\mathbf{u}$ on the right appear in a perfect square. We obtain

$$\int nm\sigma^2\mathbf{u}^2 \, dV = -\int \{P_\gamma[(\nabla \cdot \mathbf{u}) + \mathbf{u} \cdot (\nabla P_T + \mathbf{c}B^2/\mu_0)/P_\gamma]^2$$
$$+ 2\mathbf{u} \cdot \mathbf{c}[\mathbf{u} \cdot \nabla B^2/2\mu_0 + \mathbf{u} \cdot \mathbf{c}B^2/\mu_0]$$
$$- [\mathbf{u} \cdot (\nabla P_T + \mathbf{c}B^2/\mu_0)]^2/P_\gamma + (m\mathbf{g} \cdot \mathbf{u})(\mathbf{u} \cdot \nabla n)\} \, dV \qquad (20)$$

After further algebra and substitution of the equilibrium condition, (1), we can recast the equation in the following form:

$$\int nm\sigma^2\mathbf{u}^2 \, dV = -\int \{P_\gamma[(\nabla \cdot \mathbf{u}) + \mathbf{u} \cdot (\nabla P_T + \mathbf{c}B^2/\mu_0)/P_\gamma]^2$$
$$- \mathbf{u} \cdot (\nabla p - \nabla p_{cr})[2B^2/\mu_0(\mathbf{u} \cdot \mathbf{c}) + nm\mathbf{g} \cdot \mathbf{u}]/P_\gamma$$
$$+ m\mathbf{g} \cdot \mathbf{u}(\mathbf{u} \cdot \nabla n - \mathbf{u} \cdot \nabla n_{cr})\} \, dV \qquad (21)$$

where the "critical" presure and density gradients, p_{cr}, n_{cr}, are defined such that

$$\nabla p_{cr} = \gamma p[(\nabla B)/B + \mathbf{c}] \qquad (22)$$

and

$$\nabla n_{cr} = n[(\nabla B)/B + \mathbf{c}] \qquad (23)$$

Now consider the expression (21). For stability, there must be no perturbation possible for which the right-hand side of (21) is positive. The first term is a square and thus is always stabilizing; the worst case is a perturbation for which it is zero, i.e., for which

$$(\nabla \cdot \mathbf{u}) + \mathbf{u} \cdot (\nabla P_T + \mathbf{c}B^2/\mu_0)/P_\gamma = 0 \qquad (24)$$

Inspection of equations (3) and (5) shows that condition (24) corresponds to the requirement that the total perturbation pressure (plasma plus magnetic) is balanced. It has a ready interpretation in the dynamical picture of instability. Consider equation (15). The curvature and gravity forces are the sole forces acting when the total pressure vanishes. In a field where the external (gravity or other) force, the curvature, and the pressure gradients are coplanar, i.e., act in meridians in a planetary context, the total perturbation pressure balance will encourage the fastest growth of motions that are restricted to the meridian. Driving flow out of the meridian consumes energy without releasing any and is thus energetically less efficient.

The remaining terms in (21) are proportional to the square of the amplitude of the displacement. Consider the pressure gradient terms. The worst case from the point of view of stability is when the gradient in pressure points in the same direction as the curvature vector, $\mathbf{c}$, and exceeds the critical pressure gradient given by (22) over some part of the flux tube. The final terms involve gravity. Evidently, a gradient in density in the opposite direction to the gravity field reduces the stability of the system.

In the context of planetary magnetospheres it is reasonable to assume that the gravity field points in the same sense as the field curvature vector (toward the planet). It follows that normally the gravitational force will stabilize motions where the field curvature is destabilizing.

ROTATING SYSTEMS

In a rotating system the centrifugal force enters the basic equations as an effective gravitational force that points away from the planetary rotation axis. In addition, in a rotating system, Coriolis force will enter the dynamics. However, there are some further subtleties. If the system is rotating, there must be some force acting to maintain the rotation. In the case of a magnetosphere in steady state the field stresses act to enforce corotation with the planetary ionosphere. There may thus be "hidden" forces to be accounted for when one considers interchange motions in a rotating system.

We shall assume that rigid corotation is maintained on the time scale of the interchange displacements we consider. (Other possible assumptions are examined in the discussion section later in the paper.) As a result, angular momentum will

not be conserved when plasma moves at right angles to the rotation axis. The forces in the system that counter the Coriolis force in the azimuthal sense are transmitted through the plasma by the field. The energy associated with spinning flux tubes up or down is provided by the ionosphere of the planet.

With the assumption of rigid corotation there is no azimuthal perturbation velocity. In the meridian there is no Coriolis force component, and the perpendicular equation of motion takes the form

$$nm\sigma\mathbf{u} = -\nabla(p^{(1)} + BB^{(1)}/\mu_0) + 2\varepsilon BB^{(1)}/\mu_0$$
$$+ n^{(1)}m\mathbf{g}_\perp - n^{(1)}m[\mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r})]_\perp \qquad (25)$$

where $\mathbf{\Omega}$ is the angular velocity of the plasma.

A modified energy equation which replaces (13) is obtained by dotting (25) with $\mathbf{u}$. It is evident that the set of equations governing the instability remain unchanged if $\mathbf{g}$ is replaced by an effective acceleration:

$$\mathbf{g}_E = \mathbf{g} - \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}) \qquad (26)$$

In an idealized rotating system with straight field lines [*Cheng*, 1985], one or the other of the terms in (26) is usually dominant. In magnetospheric applications, although the centrifugal acceleration may dominate near the equatorial portion of a flux tube, it may be necessary to retain both terms of $\mathbf{g}_E$ because of the importance of the gravitational contribution near the feet of the flux tube.

Before proceeding, let us discuss the extreme cases where curvature or gravity clearly dominate.

INTERCHANGE MOTION IN CURVED FIELD GEOMETRY
(NO GRAVITY)

Dropping the terms involving $\mathbf{g}$ in equation (21), we have

$$\int nm\sigma^2\mathbf{u}^2 \, dV = - \int \{P_\gamma[(\nabla \cdot \mathbf{u}) + \mathbf{u} \cdot (\nabla P_T + \mathbf{c}B^2/\mu_0)/P_\gamma]^2$$
$$- 2\mathbf{u} \cdot (\nabla p - \nabla p_{cr})(B^2/\mu_0)[(\mathbf{u} \cdot \mathbf{c})/P_\gamma]\} \, dV \qquad (27)$$

We can use (27) to derive necessary and sufficient conditions for stability and thus also for instability. The worst case perturbation, i.e., the most stringent test of stability, is one for which the terms forming a complete square are zero. Therefore, as proposed above, we choose $\mathbf{u}$ such that

$$(\nabla \cdot \mathbf{u}) + \mathbf{u} \cdot (\nabla P_T + \mathbf{c}B^2/\mu_0)/P_\gamma = 0 \qquad (28)$$

or, using the equilibrium condition (1),

$$(\nabla \cdot \mathbf{u}) + 2(\mathbf{u} \cdot \mathbf{c}B^2/\mu_0)/P_\gamma = 0 \qquad (29)$$

The remaining terms make a negative contribution to the right-hand side of (21) wherever

$$(\mathbf{u} \cdot \mathbf{c})\mathbf{u} \cdot (\nabla p - \nabla p_{cr}) < 0 \qquad (30)$$

Condition (30) is a sufficient but not a necessary condition for stability. It may be satisfied over only a section of a flux tube, and yet in an interchange motion the flux tube must move as a whole. For a necessary and sufficient condition for stability we must derive a condition that holds for a tube as a whole. Let us break up the integral (27) by flux tube and reintroduce the flux tube aligned coordinate system defined following equation (6). Then using (29), one may rewrite the residual terms in (27), integrated along any given flux tube, as

$$\int nm\sigma^2 u^2 \, dx_1 = - \int dx_1 \, \mathbf{u} \cdot (\nabla p - \nabla p_{cr}) \, \nabla \cdot \mathbf{u} \qquad (31)$$

Now recall our earlier demonstration that $\nabla \cdot \mathbf{u}$ is independent of position along the flux tube. It may thus be moved outside the integral. Similarly, $\mathbf{u} \cdot (\nabla p)$ can be removed from the integral because p is constant on each flux tube, and using the local coordinates introduced earlier,

$$\mathbf{u} \cdot (\nabla p) = (u_2/h_2)\partial p/\partial x_2 \qquad (32)$$

Let us choose the coordinate x_2 to increase in the direction of curvature. The condition for stability then becomes

$$\partial p/\partial x_2 < \left[\int dx_1 \, (h_2)^{-1}(\partial p_{cr}/\partial x_2)\right]\bigg/\left(\int dx_1\right) \qquad (33)$$

Equation (33) can be further simplified once it is noted that ∇p_{cr} is related to the local gradient in flux tube volume. Consider an elemental length, ds, along the field. The local gradient of the volume, ds/B (or dx_1), in the direction of $\hat{\mathbf{e}}_2$ is given by

$$\hat{\mathbf{e}}_2 \cdot \nabla(ds/B) = ds \, \hat{\mathbf{e}}_2 \cdot [-(\nabla B)/B^2 - \mathbf{c}/B] \qquad (34)$$

and thus (33) can be rewritten as

$$\partial p/\partial x_2 + \left[\gamma p \, \partial\left(\int dx_1\right)\bigg/\partial x_2\right]\bigg/\left(\int dx_1\right) < 0 \qquad (35)$$

Writing the local flux tube volume, $\int dx_1$, as V, it follows from (35) that the necessary and sufficient condition for stability is

$$\partial(pV^\gamma)/\partial x_2 < 0 \qquad (36)$$

Now provided the curvature vector retains the same sense with respect to the pressure gradient at all points of the flux tube, as is likely to hold in any planetary magnetospheric context, one may also write the condition as

$$\mathbf{c} \cdot \nabla[pV^\gamma] < 0 \qquad (37)$$

Condition (37) is equivalent to that given by *Gold* [1959] for the low plasma pressure (low β) case. Our calculation is valid for any value of the ratio of plasma to field pressure.

LOW-PRESSURE GRAVITY (CENTRIFUGAL) DRIVEN
INSTABILITY

Let us now consider a circumstance where the plasma pressure is slight and the equilibrium force balance perpendicular to the field is maintained by an effective gravity. An example is the rapidly rotating low-pressure Io torus in the Jovian magnetosphere, where the centrifugal force is much more important than the pressure.

If the pressure terms are dropped from equation (21), it reduces to the form

$$\int nm\sigma^2 \mathbf{u}^2 \, dV = - \int [\{B^2/\mu_0\{\nabla \cdot \mathbf{u} + \mathbf{u} \cdot [(\nabla B)/B + \mathbf{c}]\}^2$$
$$+ (m\mathbf{g} \cdot \mathbf{u})\{(\mathbf{u} \cdot \nabla n) - n\mathbf{u} \cdot [(\nabla B)/B + \mathbf{c}]\}] \, dV \qquad (38)$$

Just as previously, the worst case perturbation is one where the first term forming the complete square vanishes, which here requires

$$\nabla \cdot \mathbf{u} + \mathbf{u} \cdot [(\nabla B)/B + \mathbf{c}] = 0 \qquad (39)$$

a condition requiring that the field strength is not perturbed (cf. equation (5)).

The remaining terms contribute to stability as long as

$$\int (m\mathbf{g} \cdot \mathbf{u})\{(\mathbf{u} \cdot \nabla n) - n\mathbf{u} \cdot [(\nabla B)/B + \mathbf{c}]\} \, dV > 0 \qquad (40)$$

for all possible perturbations. Let us assume that the worst case is where the displacement is in the x_2 direction and as before break up the volume integral by flux tube. For a given flux tube to be stable one requires that

$$\int (mg_2 u)\{(\mathbf{u} \cdot \nabla n) - n\mathbf{u} \cdot [(\nabla B)/B + \mathbf{c}]\} \, dV > 0 \qquad (41)$$

where g_2 is the component of $\mathbf{g}$ in the $\hat{\mathbf{e}}_2$ direction.

Recalling that for an interchange motion u_2/h_2 is constant for along the field, we may remove $\mathbf{u}$ from the integral. The stability condition for a given flux tube then requires that for all possible directions, x_2,

$$\int (mg_2 h_2)\{(h_2(\partial n/\partial x_2) - n[(\partial B/\partial x_2)/B + c_2 h_2]\} \, dx_1 > 0 \qquad (42)$$

where c_2 is the component of the curvature in the x_2 direction.

Now the presence of the gravity field implies that in general there will be a variation of the density along the flux tube. It follows that the stability condition will normally take the flux tube integral form given in (42). However, if the force field varies little along that region of the flux tube occupied by the plasma, one may rewrite the condition in the form

$$\int \{(\partial n/\partial x_2) - n[(\partial B/\partial x_2)B + c_2 h_2]\} \, dx_1 > 0 \qquad (43)$$

i.e., from (34)

$$\partial\left(\int dx_1 \, n\right)\bigg/\partial x_2 > 0 \qquad (44)$$

i.e., the total flux tube content increases in the direction of the gravitational force.

GENERAL STABILITY CRITERION FOR CURVED FIELDS

The treatments of the preceding two sections can be combined to give a general stability condition. Once the perturbation has been chosen such as to conserve total pressure, the residual terms of (21) are

$$\int nm\sigma^2 u^2 \, dV = \int \{\mathbf{u} \cdot (\nabla p - \nabla p_{cr})[2(B^2/\mu_0)(\mathbf{u} \cdot \mathbf{c}) + nm\mathbf{g} \cdot \mathbf{u}]/P_\gamma$$

$$- m\mathbf{g} \cdot \mathbf{u}(\mathbf{u} \cdot \nabla n - \mathbf{u} \cdot \nabla n_{cr})\} \, dV \qquad (45)$$

Once more we seek a flux tube integrated condition for stability that is both necessary and sufficient. As before, we shall define the $\hat{\mathbf{e}}_2$ direction as that of the locally most unstable perturbation. The amplitude of $\mathbf{u}$ can be removed from the integral by recalling that it is proportional to h_2. Similarly, gradients can be removed by noting that $\mathbf{u} \cdot \nabla$ can be replaced by $(u_2/h_2) \, \partial/\partial x_2$. Equation (28) still holds, as in the zero gravity case, and it follows that the term

$$[2(B^2/\mu_0)(\mathbf{u} \cdot \mathbf{c}) + nm\mathbf{u} \cdot \mathbf{g}]/P_\gamma$$

is equal to $(-\nabla \cdot \mathbf{u})$ and can be taken outside the integral along the tube.

The general stability requirement is then that for all possible directions the following condition holds:

$$\int \{K_2(\partial p/\partial x_2 - \partial p_{cr}/\partial x_2)$$

$$- mg_2 h_2(\partial n/\partial x_2 - \partial n_{cr}/\partial x_2)\} \, dx_1 < 0 \qquad (46)$$

where the quantity K is constant along a flux tube and is given by

$$K_2 = [2(B^2/\mu_0)(h_2 c_2) + nmh_2 g_2]/P_\gamma \qquad (47)$$

One cannot in general take any of the pressure gradient or density gradient terms out of the flux tube integral, since the pressure and density will not be uniform along the field but will be distributed such as to balance the external gravity or centrifugal force component along $\mathbf{B}$. Where the gravity or effective gravity does not vary strongly in the region in which the plasma is trapped, one may write a condition using flux tube averaged pressure and total flux tube content. Ignoring the variation of the gravity field along the tube, one has

$$K_2(\partial p/\partial x_2 - \partial p_{cr}/\partial x_2) - mg_2 h_2(\partial n/\partial x_2 - \partial n_{cr}/\partial x_2) < 0 \qquad (48)$$

where V is the flux tube volume. One may rearrange the stability requirement as

$$K_2[\partial(pV^\gamma)/\partial x_2]/V^\gamma - (mg_2 h_2/V)[\partial N/\partial x_2] < 0 \qquad (49)$$

where N is the total flux tube content, $\int dx_1 \, n$.

Condition (49) is relevant to the ring current impoundment of cold Io torus material diffusing outward in the Jovian magnetosphere. Centrifugally driven interchange motion will be inhibited by an adverse gradient of hot plasma [Siscoe et al., 1981]. Near the equator the appropriate direction for the x_2 direction is radially outward along the centrifugal force direction. In the absence of hot plasma, interchange instability will set in (i.e., interchange will spontaneously occur) when the total flux tube content N increases inward. The presence of the hot plasma modifies that condition substantially wherever there is an outward gradient of the quantity (pV^γ), in particular, as will occur at an inner boundary of the hot distribution.

THE CASE OF STRAIGHT FIELD LINES

The special case of the stability of a plasma embedded in a horizontally stratified horizontal magnetic field supported by field and gas pressure gradients against a vertical gravitational field, $\mathbf{g} = -g\hat{\mathbf{z}}$, has received considerable attention. The standard results, as first derived by Tserkovnikov [1960], can be recovered by setting $\mathbf{c}$ to zero in equation (45). The volume integrations can be dropped. One finds that stability requires

$$P_\gamma[dn/dz - (n/B)dB/dz] > n\{dp/dz - (\gamma p/B)dB/dz\} \qquad (50)$$

By algebraic manipulation and substitution from the equilibrium condition (1), equation (50) may be reduced to either of the familiar forms required for stability:

$$P_\gamma(dn/dz)/n > dP_T/dz \qquad (51)$$

$$P_\gamma(dn/dz)/n > -nmg \qquad (52)$$

with $g > 0$ [Tserkovnikov, 1960; Newcomb, 1961; Cheng, 1985].

The pertinence of these criteria was challenged by Newcomb [1961], who pointed out that the condition for interchange instability of straight field lines was a singular case in the sense that there always exists a lower threshold for noninter-

change motions. In particular, he introduced the notion of quasi-interchange motions whose stability condition was

$$-\gamma p(dn/dz)/n > nmg \qquad (53)$$

Equation (53) has the same form as (51) with the field set to zero and is clearly a more stringent stability requirement. We sketch a derivation of the result in the appendix.

Comparing equations (A2) and (A3) for the particle and magnetic pressure perturbations with the conditions (A5) and (A6) leads one to the conclusion that the dynamical requirement that the total pressure be constant constrains quasi-interchanges also. Equation (A6) can be rewritten as

$$\mathbf{V} \cdot \mathbf{v} = \partial v_\parallel/\partial s + \mathbf{V} \cdot \mathbf{u} = -\mathbf{u} \cdot \mathbf{V}P_T/\gamma p \qquad (54)$$

Using (A5), one finds

$$\gamma p \, \partial v_\parallel/\partial s = -\mathbf{u} \cdot \mathbf{V}P_T = -mng \cdot \mathbf{u} \qquad (55)$$

The left-hand term is proportional to the rate of change of internal plasma energy, and the right-hand side shows that it equals the rate of change of gravitational energy. Thus the introduction of parallel motion allows release of internal energy without requiring field compression (cf. (A5)).

Newcomb [1961] shows that if the true interchange stability condition (52) is not satisfied, then pure interchange motions grow, and quasi-interchange motions do not. However, in the regime for which (52) is satisfied but (53) is not, quasi-interchange motion will occur.

The case of straight field lines is, however, a special case. Quasi-interchange motions require very long parallel scale lengths $(L_\perp/L_\parallel \ll 1$; see discussion following (A1)). In magnetospheric applications the parallel scale length associated with gravitational effects is of the same order as the radius of curvature, $R_c = |c|^{-1}$, and the perpendicular scale length is that characteristic of the background field or plasma pressure. In these circumstances the condition for quasi-interchange is

$$R_c \gg L_B \qquad (56)$$

but one expects $R_c = L_B$. Thus we feel that the Newcomb modification and the straight field line result itself are of limited relevance to magnetospheric physics applications.

CONCLUSIONS

In this paper we have derived the conditions for interchange instability covering any value of field to plasma pressure, the effect of body forces proportional to the density (gravity, centrifugal), and the full effects of field curvature. Many of our results have been derived previously but have not been presented in the collected form that we have here.

All of the results presented for curved field geometry have an integral form. This is inevitable, and it does preclude making hard and fast judgments from parameters measured at one point. The essential feature of an interchange motion is that flux tubes move as wholes. The energetics of such motions evidently depend on the entire flux tube population.

Recognition that plasma conditions may vary with respect to position along the field line will be particularly important when density dependent gravitational or centrifugal forces are present. In such circumstances the equilibrium forces have components parallel to the magnetic field, and there may be a strong variation in the background parameters parallel to the flux tube. Furthermore, as an interchange motion proceeds, the equilibrium distribution parallel to the magnetic field di-

rection will be disturbed, and the motion will be accompanied by transient processes redistributing plasma along the field.

In a rotating system there is further complication, because radial motion requires consideration of the redistribution of angular momentum along flux tubes as interchange takes place. The question has been discussed by Newcomb [1962], who pointed out that magnetic stresses in any particular flux tube would redistribute angular momentum along the tube. In circumstances where, in the absence of the field, the angular momentum of fluid elements would be conserved, the redistribution brings tubes to a constant angular velocity, and thus it is the mean angular momentum in the tube that is constant.

Recently, the problem of stability of a rotating system has been reexamined by Rogers and Sonnerup [1986], who invoke conservation of angular momentum to add a further term to an energy expression similar to (21) and others in this paper. There is an important difference between the plasma configurations considered by Newcomb [1962] and Rogers and Sonnerup [1986] and what is appropriate for a planetary magnetosphere. All flux tubes in a magnetosphere attach to a conducting ionosphere, which in turn rotates with the planet. We have assumed that the energy to maintain corotation is provided by the planet and thus does not enter our calculations. As can be deduced from the nonconservation of angular momentum, there must be forces introduced in the process of the interchange that are azimuthal (noncentral) in order to keep tubes rigidly rotating. Evidently, as one moves away from the planet, the time scale for imposing corotation increases. There is a case for further work to consider departures from rigid rotation. The work of Rogers and Sonnerup [1986] shows that the Coriolis force must be carefully considered once the system departs from rotation at a single angular velocity. Its effect is to make interchange motions more stable than in the rigidly rotating system. It follows from their work that the standard energy principle for a nonrotating system, modified only by inclusion of an effective centrifugal gravity (i.e., as given in equation (21)), yields a sufficient condition for stability.

In magnetospheric applications the rotational and the gravitational forces are often not important, and the curvature effects dominate. Our results show that in this situation the interchange instability condition depends on how the pressure varies with flux tube volume, V. The plasma is unstable, and one expects interchange motions to be spontaneously generated, when the pressure in the plasma decays more slowly with flux tube volume than if it were distributed according to the law

$$pV^\gamma = \text{const} \qquad (57)$$

A distribution satisfying (57) would be set up if the plasma were injected from regions of large flux tube volume, i.e., high L shell, to small flux tube volumes at low L by a process wherein the pressure was maintained isotropic with respect to the field and there was negligible loss. Once loss becomes important in any such flow, the pressure gradient drops below the "adiabatic" value given by (57), and the distribution would be stable. In practice there is a little difference in general principle when there is no systematic isotropization of the plasma as the plasma moves to smaller volume flux tubes. Anisotropy in the pressure results in nonuniform plasma distribution along the field line and doubtless differences in the details of the precise condition. However, the general orders of

magnitude implied by the condition (57) are sound unless a very extreme distribution is chosen.

Recent controversy about the effect of the ring current pressure gradient on the rate of net outward motion (or diffusion) of Iogenic plasma through interchange instability can now be resolved. The flux tube content, N, of the Iogenic torus plasma has a negative gradient outside the orbit of Io at 6 R_J (R_J is the radius of Jupiter). That gradient changes markedly in the vicinity of flux tubes crossing the equator at 8 R_J. Siscoe et al. [1981] proposed that near 8 R_J the relatively dense and cool torus plasma is "impounded" by a steep outward gradient of low-density energetic ring current plasma, which inhibits its further outward diffusion. Cheng [1985] challenged this description on the grounds that in the presence of strong centrifugal effects the positive pressure gradient would be interchange unstable. His argument appeals to the instability condition for straight field line geometry (equation (51) or (52)). Reversing the inequality to apply to the case of an outward directed centrifugal force, one finds that when field lines are straight, a negative density gradient and a positive pressure gradient are necessarily unstable to interchange.

As we have shown, all is changed once the consequences of field curvature are considered. The relevant condition for instability is that the right side of (49) must be positive. A requirement for instability in the latter expression is that the gradient in pressure must be directed toward the center of curvature of the field line, i.e., inward. Thus the outward pressure gradient of the ring current would inhibit interchange motion. Cheng's [1985] criticism of the conclusions of Siscoe et al. [1981] concerning the stabilizing effect of an outward pressure gradient does not hold once curvature effects are included (as Cheng himself points out may be so).

There have been papers such as the one by Newcomb [1961] or, recently, Viñas and Madden [1986] which have pointed out that in addition to interchange there exist similar instabilities wherein the field is allowed to bend. Under some circumstances, such instabilities may have thresholds lower than the interchange. The Viñas and Madden paper emphasizes shear ballooning instabilities, but the treatment covers some interchange motions also. Ballooning motions are very important and may well give rise to wave and wavelike structures in space plasmas. Some fraction of the large variety of magnetic pulsation phenomena reported in the earth's magnetosphere can doubtless be attributed to such effects. However, the interchange motion has a distinctive property that singles it out for attention; it is the sole class of motion that leaves the basic field configuration unchanged, and thus it can potentially describe steady configurations wherein plasma created within the system or injected from the solar wind or planetary ionosphere is redistributed by a self-induced or driven convection.

There may of course be field and plasma configurations in which interchange motions are not possible. Such a field would not be able to support a steady plasma adiabatic convection system, and plasma redistribution would be an inherently unsteady process. For example, it has been proposed that the earth's magnetotail has such a property [Erickson and Wolf, 1980].

In the treatments presented here, many potential effects have been neglected. In particular, a kinetic approach can be very useful. When plasma pressure is small compared with the field pressure, one expects that a marginally stable distribution will correspond to one resulting from the loss free injection of

particles from large flux tube volume regions to small under conservation of the first two adiabatic invariants. The condition in such a case is closely related to the minimum-B plasma equilibrium condition [Taylor, 1964] and was considered in detail by Chang et al. [1965].

Another extremely important feature of a kinetic or quasi-kinetic treatment (see, for instance, Richmond [1973]) is that the wavelength or scale length of the flux tube motions perpendicular to the direction of inhomogeneity becomes important. As Richmond [1973] shows, an unstable distribution of low-energy plasma may be capable of executing an interchange motion in the presence of a stabilizing high-energy (i.e., ring current) population, if the energy difference of the distributions is significant enough. The decoupling is accomplished on short scales because the more energetic particles ∇B drift through the small-scale structures fast enough that their integrated displacement is much smaller than that of the lower energy unstable particles.

In a system where rotation is unimportant, the ionosphere cannot affect the actual instability condition. It behaves very like a frictional boundary layer, and it does control the rate at which any motion takes place (see, for example, Chang et al. [1965] and Richmond [1973]). As we have discussed above, in a rotating system, matters are more complicated, and the energetics are affected by the degree to which the ionosphere can impose corotation on the magnetospheric flux tube.

APPENDIX: QUASI-INTERCHANGE MOTIONS IN A STRAIGHT FIELD GEOMETRY

Newcomb's result, given as equation (53) above, is found by relaxing the constraint that there be no variation in the perturbation along the direction of the background field. Quasi-interchange motions allow for variation to first order along the background field. Then the perturbation magnetic field must be allowed to have perpendicular as well as parallel components. In addition, we allow for parallel components of the flow.

If we drop the background field curvature, the first-order equation replacing (14) becomes

$$nm\sigma v^2 = -v \cdot \nabla p^{(1)} - u \cdot \nabla B \, \delta B_\parallel / \mu_0)$$
$$+ v \cdot [\delta B_\perp \cdot \nabla B / \mu_0] + n^{(1)} mg \cdot u \qquad (A1)$$

where $v = (u, v_\parallel)$ and $B^{(1)} = (\delta B_\perp, \delta B_\parallel)$. Note that the familiar "ballooning" term ($B \cdot \nabla \delta B_\perp$) [cf. Viñas and Madden, 1986] is dropped in the quasi-interchange approximation; the term is smaller than the other new terms in (A1) by a factor of $L_\perp/L_\parallel$ (the ratio of characteristic scale lengths perpendicular and parallel to the unperturbed field).

The pressure and density perturbations are modified by the possibility of moving plasma along the field. Thus equation (3) is replaced by

$$\sigma p^{(1)} = -\gamma p (\nabla \cdot v) - u \cdot \nabla p \qquad (A2)$$

and equation (2) is similarly modified. Equation (5) for the magnetic pressure perturbation is unaffected by parallel motion:

$$\delta B B^{(1)} / \mu_0 = -(\nabla \cdot u) B^2 / \mu_0 - u \cdot \nabla B^2 / 2\mu_0 \qquad (A3)$$

The manipulations proceed as in the interchange calculation, and the generalization of equation (21) is

$$\int nm\sigma^2 \mathbf{u}^2 \, dV = -\int \{(B^2/\mu_0)(\nabla \cdot \mathbf{u})^2 + \gamma p[\nabla \cdot \mathbf{v} + \mathbf{u} \cdot \nabla P_T/\gamma p]^2$$

$$-(\mathbf{u} \cdot \nabla P_T)^2/\gamma p + m\mathbf{g} \cdot \mathbf{u}(\mathbf{u} \cdot \nabla n)\} \, dV \qquad \text{(A4)}$$

The worst case for stability occurs when

$$\nabla \cdot \mathbf{u} = 0 \qquad \text{(A5)}$$

and

$$\nabla \cdot \mathbf{v} + \mathbf{u} \cdot \nabla P_T/\gamma p = 0 \qquad \text{(A6)}$$

Stability then requires that

$$\mathbf{g} \cdot (\nabla n - n\nabla P_T/\gamma p) > 0 \qquad \text{(A7)}$$

This is directly equivalent to the statement (53) of the text.

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M. G. Kivelson, Department of Earth and Space Sciences, 3806 Geology Building, University of California, Los Angeles, CA 90024.

D. J. Southwood, Institute of Geophysics and Planetary Physics, 3839 Slichter Hall, University of California, Los Angeles, CA 90024.