Reviews of Geophysics and Space Physics, Vol. 9, No. 3, August 1971

Plasma Instabilities in the Magnetosphere

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An introductory review of theories of plasma instabilities in the magnetosphere is presented. Part *A* is a review of theories of plasma instabilities that are relevant to magnetospheric plasmas. Instabilities arising from velocity-distribution anisotropies, such as a pitch-angle anisotropy or the presence of beams, as well as instabilities from nonuniform distributions of plasmas and magnetic fields, are discussed. Particular emphasis is placed on the effect of a mixture of a cold plasma in a high-β plasma. Part *B* is a summary of works related to actual plasma instabilities in the magnetosphere. In view of the observed plasma parameters, it is shown that the magnetosphere is rather stable against most macroscopic instabilities, and its dynamics are predominantly governed by microscopic instabilities.

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1. INTRODUCTION

The magnetosphere is filled with almost-ideal plasma composed almost purely of protons and electrons. The Debye length corresponding to the low-energy electrons ($\sim$100 meters) is small compared with the scale size, but there exist enough particles per Debye sphere ($\gtrsim 10^{12}$) for those particles to behave collectively as a plasma. Collision effects, even for cold electrons inside the plasmapause, are negligible in most cases because the collision frequency there is of the order of 10^{-5}/sec–10^{-4}/sec. Recent measurements have revealed that the plasma is composed of three distinct energy groups: the medium- to high-energy group (>100 kev), the low-energy group ($\sim$kev), and the cold or thermal group ($\sim$ev). Most of the collective effects are governed by the low-energy and cold-plasma groups. The low-energy group contains the largest energy density. During quiet times, its energy density is approximately 10% of the energy density of the background geomagnetic field; during active times the energy density exceeds that of the magnetic field. Hence the ratio β of plasma pressure to magnetic-field pressure, an important parameter in studying plasma instabilities, must be regarded as comparable to unity.

Papers published on instabilities of the magnetospheric plasma now number easily more than a hundred. However, many of these theories need revision because of more recent discoveries of plasma characteristics in the magnetosphere. For example, the high β effect, which has been ignored in much previous work, needs to be taken into account.

Direct observations of electric and magnetic fields ranging from a frequency of several kilohertz to a period of several minutes are being made simultaneously with measurements of the plasma characteristics (such as density, energy, anisotropy, etc.). These observations indicate that plasma instabilities are responsible for many of the interesting dynamic phenomena observed in the magnetosphere.

Only linear instabilities are treated throughout the paper, except in chapter 5. However, the nonlinear effects and instabilities are by no means negligible. The elimination of these effects from the present paper is simply for convenience.

A. Review of Theory of Plasma Instabilities

2. INTRODUCTION TO PLASMA INSTABILITIES

2.1. *Dispersion Relations*

A general concept of plasma instability is presented here. It covers the definition of instability and several 'know-hows' of finding and analyzing the instabilities.

As in most cases, the general concept is more easily understood by considering a typical example. For example, consider a two-stream instability in a system of an electron stream with velocity v_0 and a stationary plasma. If we restrict ourselves to a frequency range much higher than the ion plasma frequency, we can ignore ion dynamics. (We shall discuss the case where ion dynamics are involved in chapter 3).

The basic equations that describe plasma dynamics are Maxwell's equations, which give the electromagnetic field produced by the current and charge distributions, and the equations of motion, which describe the motions of charged particles in the electromagnetic fields. For the equations of motion, one can use the MHD (magneto-hydrodynamic) equations, the Vlasov equation, or a simple single-particle equation of motion, depending on the nature of the problem. The MHD equations, to be introduced in subsection 3.2*d*, represent a fluid approximation of plasma dynamics. Although they are often simpler to handle than the Vlasov equations, they become invalid when particle dynamics are important. The Vlasov equation, to be introduced in subsection 3.1*a*, is usually more difficult to solve but is accurate in describing particle dynamics in the absence of collisions. Collision effects, if important, have to be considered in a suitable way depending on the nature of the problem (see, for example, subsection 3.1*e*).

In the example we are considering here, if the electron temperature is high so that the thermal velocity v_T is comparable with or larger than v_0 or the phase velocity of the wave v_p, we have to use the Vlasov equation. However, if we consider a cold electron stream in a cold plasma ($v_T = 0$), a single-particle equation of motion is sufficient. Here the electron stream and the cold plasma form the two streams.

We first consider the dynamics of the stream electrons. The equation of motion is

$$m_e(d\mathbf{v}/dt) = -e(\mathbf{E} + \mathbf{v}\times\mathbf{B}) \tag{2.1}$$

where $\mathbf{v}$, m_e, e, $\mathbf{E}$, and $\mathbf{B}$ are velocity, electron mass, electron charge, electric field intensity, and magnetic-flux density, respectively. Because we do not know beforehand the kind of electromagnetic field produced by the stream electrons, we have to include arbitrary fields in the equation of motion.

Because the electron stream is moving with respect to the stationary frame of the plasma, it is convenient to write the total derivative in equation 2.1 in partial derivatives in time and space, i.e.

$$\frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} \tag{2.2}$$

where

$$\mathbf{v}\cdot\nabla = v_x\frac{\partial}{\partial x} + v_y\frac{\partial}{\partial y} + v_z\frac{\partial}{\partial z} \tag{2.3}$$

When one is concerned with the linear instability, the standard procedure at this point is the linearization. We write the dependent quantities in terms of dc or slowly varying quantities (subscript 0) and rapidly varying quantities (subscript 1). For example, for the velocity of the electron stream v

$$\mathbf{v} = \mathbf{v}_0 + \mathbf{v}_1(\mathbf{x}, t) \tag{2.4}$$

We consider the quantity with subscript 1 to be a perturbation to the state represented by subscript 0. Hence $|\mathbf{v}_0| \gg |\mathbf{v}_1|$.

Equation 2.1 for order zero is then

$$\frac{\partial \mathbf{v}_0}{\partial t} + (\mathbf{v}_0 \cdot \nabla)\mathbf{v}_0 = -\frac{e}{m_e}(\mathbf{E}_0 + \mathbf{v}_0 \times \mathbf{B}_0) \tag{2.5a}$$

If the stream is stationary in time and spatially uniform, $\partial v_0/\partial t$ as well as $\partial v_0/\partial x$ vanish. This means that we are considering an infinitely extended stream of electrons. Such a scheme is allowed if ions are also moving with the same velocity v_0 making the total current vanish. Then it is immediately obvious that v_0 can have an arbitrary magnitude parallel to B_0 or E_0/B_0, perpendicular to B_0. The perpendicular velocity produced by a dc electric field E_0 should be the same for both the stream and the stationary electrons; hence in the frame of E_0/B_0 there will be no streaming perpendicular to B_0. (The $\mathbf{E}_0 \times \mathbf{B}_0$ drift however *does* produce two streaming between electrons and ions if the collision frequencies for these species is different, as will be discussed in subsection 3.1*e*.)

Thus in the collisionless case, two streaming is possible only in the direction parallel to $\mathbf{B}_0$. We set

$$\mathbf{v}_0 = v_0 \mathbf{e}_\parallel \tag{2.5b}$$

where $\mathbf{e}_\parallel$ is the unit vector parallel to $\mathbf{B}_0$. The zeroth-order solution is called that of the *equilibrium state*. When one considers an instability, it is very important to find the equilibrium state first. It is meaningless to discuss the instability of a state where no equilibrium state exists (exceptions are allowed for some limited cases, where the initial nonequilibrium state becomes unstable with a growth rate much faster than the rate of approach to equilibrium).

We now employ a rectangular coordinate system and take the z axis in the direction of the dc magnetic field B_0. Then the first-order equation for equation 2.1 becomes

$$\frac{\partial \mathbf{v}_1}{\partial t} + v_0 \frac{\partial \mathbf{v}_1}{\partial z} = -\frac{e}{m_e}(\mathbf{E}_1 + \mathbf{v}_0 \times \mathbf{B}_1 + \mathbf{v}_1 \times \mathbf{B}_0) \tag{2.6}$$

The next step usually taken after linearization is Fourier–Laplace transformation of the dependent variables, for example $v_1(\mathbf{x}, t)$

$$v_1'(\omega, \mathbf{k}) = \int_0^\infty dt \int_{-\infty}^\infty d\mathbf{x}\, v_1(\mathbf{x}, t) e^{i(\omega t - \mathbf{k}\cdot\mathbf{x})} \tag{2.7}$$

where $v_1(\mathbf{x}, t)$ is alternatively given by

$$v_1(\mathbf{x}, t) = \frac{1}{(2\pi)^4} \int_{-\infty+i\sigma}^{\infty+i\sigma} d\omega \int_{-\infty}^{\infty} d\mathbf{k}\, v_1'(\omega, \mathbf{k}) e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)} \tag{2.8}$$

For the stability analysis, we usually look for an instability not depending on the initial condition of the perturbations. If we set all the initial values to zero, the transformed results have identically the same form as the one obtainable by substituting a complex amplitude function defined, for example, by

$$v_1(\mathbf{x}, t) = v_1''e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} \tag{2.9}$$

Note that, however, v_1' in equation 2.7 and v_1'' in equation 2.9 have different dimensions. It is also important to remember that the transformation in time in equation 2.7 is valid for the plane Im $\omega > 0$, so that the integration over time can converge at $t \to +\infty$. This fact will be used in chapter 3. With these two points in mind, one can obtain the transformed result equivalently by substituting the amplitude function defined in equation 2.9. The result then becomes (for simplicity, we delete prime or double prime for the transformed quantity)

$$i(k_z v_0 - \omega)\mathbf{v}_1 = -\frac{e}{m_e}(\mathbf{E}_1 + \mathbf{v}_0 \times \mathbf{B}_1 + \mathbf{v}_1 \times \mathbf{B}_0) \tag{2.10a}$$

At this stage it is usually convenient to consider *the direction of wave propagation.* This is decided by the direction of the wave vector $\mathbf{k}$ used in the transformation. Here we take $\mathbf{k}$ also in the z direction; thus $\mathbf{v}_0$, $\mathbf{B}_0$, and $\mathbf{k}$ are taken parallel to each other. By doing this we lose an electrostatic cyclotron wave instability [for example, *Briggs,* 1964] and an electromagnetic instability [*Weibel,* 1959] but gain a considerable simplification, which is important for presenting an example. Then, as will be seen, the equation of continuity shows that $\mathbf{v}_1$ is directed also in z direction, whereas Poisson's equation gives $\mathbf{E}_1$ in the z direction. This means there exists no perturbed magnetic field $\mathbf{B}_1$ in this case. $\mathbf{v}_1 \times \mathbf{B}_0$ also vanishes because $\mathbf{v}_1$ is parallel to $\mathbf{B}_0$. Equation 2.10*a* then simplifies to

$$v_1 = \frac{-E_1 e/m_e}{i(kv_0 - \omega)} \tag{2.10b}$$

where v_1, E_1, and k are in the z direction.

Now we have to obtain E_1. The sources of the electromagnetic fields are current density and charge density. We have to derive them from the velocity in equation 2.10*b*. Equations needed for this are the relation of the current density J to the particle velocity v and number density n, which, in the linearized form, becomes (for an electron stream)

$$J_1 = -e(n_0 v_1 + n_1 v_0) \tag{2.11}$$

and the equation of continuity

$$-e\frac{\partial n_1}{\partial t} + \frac{\partial J_1}{\partial x} = 0 \tag{2.12a}$$

or

$$-e\omega n_1 + kJ_1 = 0 \tag{2.12b}$$

(Equation 2.11 is not needed when MHD equations are used.) By eliminating v_1 and n_1 from equations 2.10*b*, 2.11, and 2.12*b*, we can express J_1 in terms of E_1 and dc quantities as

$$J_1 = -i\omega\epsilon_0\left[\frac{-\omega_{ps}^2}{(\omega - kv_0)^2}\right]E_1 \tag{2.13}$$

where ω_{ps} is the plasma frequency of the stream electrons given by

$$\omega_{ps} = \left(\frac{e^2 n_0}{\epsilon_0 m_e}\right)^{1/2} \tag{2.14}$$

and ϵ_0 is the space dielectric constant ($=8.854 \times 10^{-12}$ F/m).

Here we introduce the concept of equivalent dielectric constant. In the second curl equation of Maxwell's equations

$$\nabla \times \mathbf{H} = \mathbf{J} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \tag{2.15a}$$

if we know the relation between the current density **J** and electric field **E**, such as shown in equation 2.13, the right-hand side may be equivalently written as

$$\nabla \times \mathbf{H} = -i\omega\epsilon_0(1 + \boldsymbol{\varepsilon})\mathbf{E} \tag{2.15b}$$

$\boldsymbol{\varepsilon}$ expressed as equation 2.16, which is a tensor in general, is called the equivalent dielectric constant (tensor). In the case of the electron stream, ϵ becomes scalar and is given from equation 2.13 by

$$\epsilon = -\frac{\omega_{ps}^2}{(\omega - kv_0)^2} \; (\equiv \epsilon_s) \tag{2.16}$$

Exactly in the same way, one can derive the equivalent dielectric constant, ϵ_p, for the stationary plasma electrons

$$\epsilon_p = -(\omega_p^2/\omega^2) \tag{2.17}$$

where ω_p is the plasma frequency of the stationary electrons. Because there exists no perturbed magnetic field, equation 2.15*b* reduces to

$$J_1 - i\omega\epsilon_0 E_1 = 0 \tag{2.15c}$$

If we use for J_1 in equation 2.15*c* the sum of the perturbed current in the stream and that in the plasma, we have

$$-i\omega\epsilon_0(1 + \epsilon_s + \epsilon_p)E_1 = 0 \tag{2.15d}$$

The nontrivial solution of equation 2.15*d* is given by

$$D(\omega, k) \equiv \omega(1 + \epsilon_s + \epsilon_p) = 0 \tag{2.18a}$$

Equation 2.18*a* is called the *dispersion relation.*

Let us now briefly review the process we took to derive the dispersion relation. We first assumed an arbitrary electromagnetic field and obtained the response in particle motion produced by the Lorentz force. We have linearized the equation by assuming the perturbation is infinitely small. We then obtained the electromagnetic field produced by the charge and the current distribution that appear in consequence of the initially assumed electromagnetic field. The dispersion relation represents a relation between ω and k that makes the assumed electromagnetic field consistent with the induced field for an infinitely small perturbation of a form $e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}$. Thus if the dispersion relation gives a root for ω with positive imaginary part, the self-consistent field grows exponentially in time.

2.2. Analysis of Instability

A plasma is called linearly unstable when the dispersion relation

$$D(\omega, \mathbf{k}) = 0 \tag{2.19}$$

has a solution for ω with a positive imaginary part for any real value of $\mathbf{k}$. What we mean by linearly unstable may be illustrated in Figure 1*a* and *b*. Fig. 1*a* is a commonly used picture of a ball sitting at the top of a hill. This case represents a linearly unstable situation because an infinitely small perturbation would kick the ball down.

Case (*a*) may be called explosively unstable because a finite displacement of the ball position does not lead to a stable situation, whereas, in contrast, in case (*b*) the ball reaches the next hill and is reflected back, hence it will be stabilized with a finite (nonzero) size of perturbation (displacement). Case (*c*), on the other hand, may look stable, but if a large enough perturbation is applied to the ball, it becomes unstable. Such a case is generally called nonlinearly unstable. Case (*d*) represents an absolute stability.

Throughout the text we will discuss situations represented either by case (*a*) or (*b*), without trying to distinguish between them. However, as can be easily seen from different cases in Figure 1, nonlinear effects are very important in studying real dynamics of a plasma. One has to bear in mind that linear instability analysis does not give a solution to all plasma dynamics.

We will now discuss the nature of the dispersion relation given in equation 2.19 in the context of stability analysis. The arguments are those developed in a recent paper by the author [*Hasegawa*, 1968]. In the particular example of a

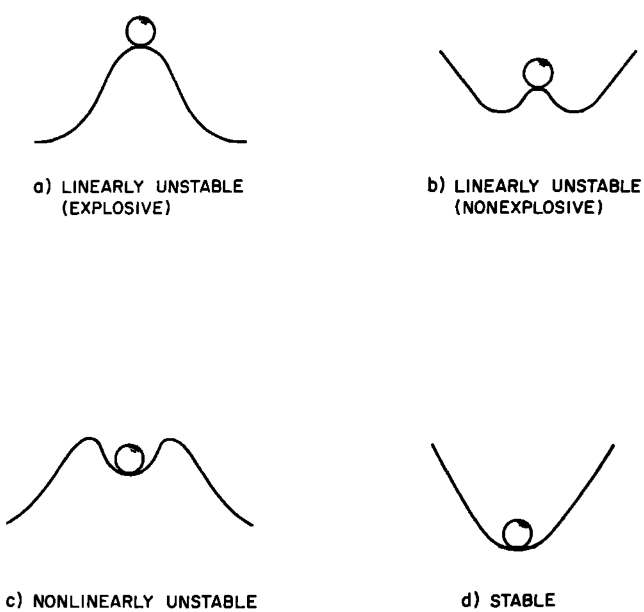

Fig. 1. Models of various stability conditions.

stream of electrons and a stationary plasma, the dispersion relation has the following form (from equations 2.16 and 2.17)

$$1 - \frac{\omega_p^2}{\omega^2} - \frac{\omega_{ps}^2}{(\omega - kv_0)^2} = 0 \tag{2.18b}$$

Although equation 2.18*b* is a fourth-order algebraic equation for ω, it can easily be seen that two of the roots can become complex by plotting the left-hand side versus ω and counting the number of zero crossings. It is also easy to see that if $v_0 = 0$, the complex roots disappear; thus the instability arises from a relative velocity between two groups of electrons.

In this section, however, we are not interested only in the fact that a two-stream flow of electrons becomes unstable. We are interested in a more general theory of the nature of the dispersion relation that leads to an instability.

The fact that the instability is generated by the ϵ_s part of the dispersion relation implies that ϵ_s has a particular nature as a dielectric constant. According to *Landau and Lifshitz* [1960], the electric field energy W of a wave propagating in a *lossless* dielectric medium can in general be expressed as

$$W = \frac{\partial(\omega\epsilon)}{\partial\omega} \frac{\langle E^2 \rangle}{2} \tag{2.20}$$

where $\langle E^2 \rangle$ is the time average of the square of the electric-field amplitude. If we use this relation, the energy of a wave in a stream W_s, and in a plasma W_p, can be calculated, respectively, as

$$W_s \propto \frac{\partial(\omega\epsilon_s)}{\partial\omega} = \frac{(\omega + kv_0)}{(\omega - kv_0)^3} \omega_{ps}^2$$

and

$$W_p \propto \frac{\partial(\omega\epsilon_p)}{\partial\omega} = \frac{\omega_p^2}{\omega^2}$$

Thus we can see that W_p is always positive, whereas W_s can be negative if $\omega < kv_0$. Now the dispersion relation for stream electrons only can be obtained from (2.18*b*) by setting $\omega_p = 0$ as

$$\omega - kv_0 = \pm\omega_{ps} \tag{2.21}$$

Therefore the mode corresponding to the lower sign of equation 2.21 (called the slower wave) does in fact satisfy the relation $\omega < kv_0$. Hence this mode carries a negative energy. Such a wave is called the negative-energy wave. The present two-stream instability can be interpreted as caused by the coupling between the negative-energy wave in the stream and the positive-energy wave in the plasma.

If the coupling is weak such that the coupling occurs between only two modes (the above example is not this case, because four waves whose dispersion relations are given by $\omega = kv_0 \pm \omega_{ps}$, and $\omega = \pm\omega_p$ are coupling simultaneously), four different cases can occur as shown in Figure 2 [*Sturrock*, 1958]. Plotted in this figure are portions of the dispersion relation in $\omega - k$ coordinates that represent couplings between two modes of different characteristics. Coupling occurs when

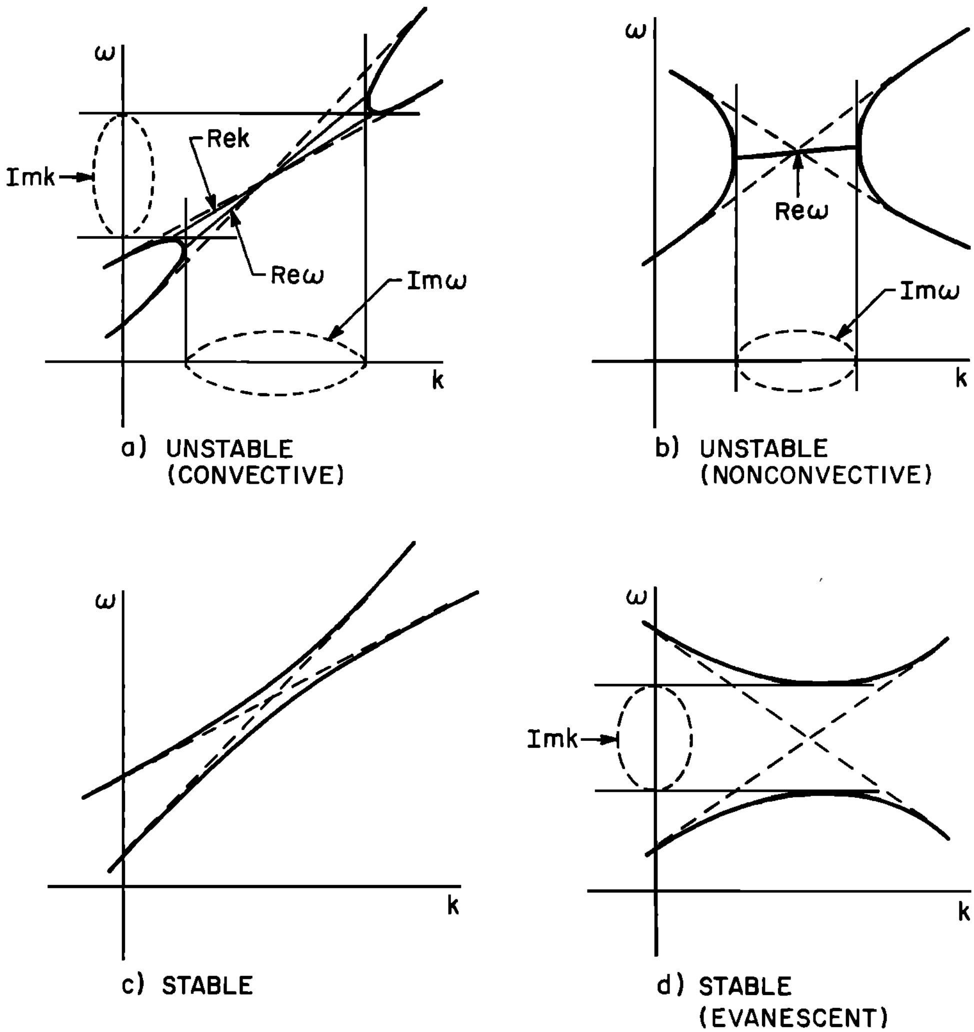

Fig. 2. Dispersion diagrams of two coupled waves of various natures: (*a*) opposite energy, same group velocity; (*b*) opposite energy, opposite group velocity; (*c*) same energy, same group velocity; and (*d*) same energy, opposite group velocity.

two modes have the same phase velocity at the same frequency ω. Thus it can be represented by the crossing of two lines that represent dispersion relations of each mode in the $\omega - k$ diagram.

Cases *a* and *b* are for two waves with opposite signs of energy, whereas *c* and *d* are for two waves with the same sign in energy. In case *a* or *b*, a complex ω solution results for real k, hence representing unstable situations; in case *c* or *d*, ω is always real, thus representing stable situations. The difference between *a* and *b* (as well as *c* and *d*) is in the sign of the group velocity ($v_g = \partial\omega/\partial k$) of the two coupling modes. Specifically, cases *a* and *c* are for the same sign in the group velocity, but *b* and *d* are for opposite signs.

In case a, k also can become complex; thus a spatial amplification of a wave at fixed real frequency ω is possible. Such a case is called convective instability. In case b, the wave number always remains real, hence the instability does not convect away. Such a case is called nonconvective or absolute instability.

It can be shown in general [for example, *Hasegawa*, 1968] that in a lossless system the necessary condition for an instability to occur (for the root of the dispersion relation to have Im $\omega > 0$) is that part of the dielectric constant of a plasma (for example, in the previous case, ϵ_s of equation 2.18a), should satisfy the condition of a negative-energy wave, i.e.

$$\partial(\omega\epsilon)/\partial\omega < 0 \tag{2.22}$$

for real ω.

We now consider a situation in which losses in the system, either due to collisions or wave-particle interactions, are not negligible. In such a case the necessary condition for an instability is that a part of the dielectric constant satisfies [*Hasegawa*, 1968]

$$\text{Re } \sigma \equiv \text{Re } (-i\omega\epsilon) = \text{Im } (\omega\epsilon) < 0 \tag{2.23}$$

for real ω, where $\sigma(= -i\omega\epsilon_0\epsilon)$ represents the equivalent conductivity of a plasma. Equation 2.23 represents simply a condition of negative real conductivity, or in other words, negative dissipation. One can produce equation 2.22 from equation 2.23 for a case with a small loss by expanding σ in powers of small Im ω (>0) and requiring Re $\sigma < 0$, hence equation 2.23 represents a more general necessary condition for an instability than equation 2.22.

We will now see how such cases arise by using again the example of two-stream electrons in two extreme cases. In the first case, the stream electrons are collision-dominated, whereas in the second case the stationary electrons are collision-dominated. Both cases produce instability, but the difference in physical mechanism is worth appreciating.

The collision effect for a cold plasma can be brought in simply by introducing a Langevin-type friction term, νv, into the equation of motion, equation 2.1, where ν is an equivalent momentum-transfer collision frequency between electrons and other species. In the first case, where the stream electrons are collision-dominated, ϵ_s is modified to

$$\epsilon_s = \frac{i\omega_{ps}^2}{\nu(\omega - kv_0)} \tag{2.24}$$

where the recombination rate is assumed to be zero. From equation 2.24, we can immediately see that when $\omega - kv_0 < 0$, ϵ_s satisfies the condition (2.23), the condition of negative dissipation. According to the theory presented, the system can still become unstable. To see this we now write down the entire dispersion relation (assuming no collisions for plasma electrons), following equation 2.18a

$$1 - \frac{\omega_p^2}{\omega^2} + \frac{i\omega_{ps}^2}{\nu(\omega - kv_0)} = 0 \tag{2.25}$$

If we assume for simplicity $\omega_{ps} \ll \omega_p$, the solution of equation 2.25 can be written as

$$\omega = \omega_p - i\,\frac{\omega_p \omega_{ps}^{\ 2}}{2\nu(\omega_p - kv_0)} \tag{2.26}$$

Hence for $k > \omega_p/v_0$, the instability results.

Now we consider the second case where the plasma electrons are collision-dominated. In this case, the dielectric constant for the plasma electrons becomes

$$\epsilon_p = i\omega_p^{\ 2}/\nu\omega \tag{2.27}$$

Naturally Im $\epsilon_p > 0$, and the dielectric constant represents a dissipative medium. The full dispersion relation now reads

$$1 - \frac{\omega_{ps}^{\ 2}}{(\omega - kv_0)^2} + \frac{i\omega_p^{\ 2}}{\nu\omega} = 0 \tag{2.28}$$

and the approximate solution is obtained for $kv_0 \sim \omega_p \gg \omega_{ps}$

$$\omega = kv_0 \pm \omega_{ps}\left(1 - \frac{i\omega_p}{2\nu}\right) \tag{2.29}$$

The lower sign that corresponds to the negative-energy wave has a solution with Im $\omega > 0$. Equation 2.29 implies that a negative energy wave can cause instability by coupling not only to a positive-energy wave but also to a dissipative medium. This is because the amplitude of the negative energy wave grows by dissipating its energy.

With these preparations, we can now derive an important general theory. If we define an equivalent longitudinal (electrostatic) plasma conductivity by

$$\sigma(\omega, k) = i\omega q n_1/k^2\varphi_1 \tag{2.30}$$

where q is the particle charge and φ_1 is the perturbed electrostatic potential defined by

$$\varphi_1 = \mathbf{E}_1/(-i\mathbf{k}) \tag{2.31}$$

and $\mathbf{k}$ is the wave number, the dispersion relation of an electrostatic mode can in general be written as [*Hasegawa*, 1968]

$$-i\omega\epsilon_0 + \sigma(\omega, k) = 0 \tag{2.32}$$

where $k = |\mathbf{k}|$.

We consider a situation such that the system has only a small loss. Mathematically this means that for a real frequency ω and wave number k, $|\text{Re}\ \sigma| \ll |\omega_r(\partial\ \text{Im}\ \sigma/\partial\omega)|$, and equation 2.3 is satisfied by the *real* frequency $\omega(=\omega_r)$ at *real* wave number $k(=k_r)$ such that

$$-i\omega_r\epsilon_0 + i\ \text{Im}\ \sigma(\omega_r, k_r) = 0 \tag{2.33}$$

The imaginary part of $\omega(=\omega_i)$ can be obtained in the following way. We expand the dispersion relation equation 2.33 around $\omega = \omega_r$ in powers of $i\omega_i$

$$-i\epsilon_0(\omega_r + i\omega_i) + i\ \text{Im}\ \sigma(\omega_r, k_r) + \text{Re}\ \sigma(\omega_r, k_r) + i\omega_i\,\frac{i\partial\ \text{Im}\ [\sigma(\omega_r, k_r)]}{\partial\omega_r} = 0 \tag{2.34}$$

Using equation 2.33, we have from equation 2.34

$$\omega_i \equiv \text{Im}\,\omega = \frac{\text{Re}\,[\sigma(\omega_r, k_r)]}{\{\partial\,\text{Im}\,[\sigma(\omega_r, k_r)]\}/\partial\omega_r - \epsilon_0} \tag{2.35a}$$

Equation 2.35a is quite a useful relation though it applies only in limited situations with small loss. We have used σ and ϵ in a mixed way because σ is more natural when we talk about loss, whereas ϵ is more natural when we talk about energy. The relation between σ and ϵ is

$$-i\omega\epsilon\,\epsilon_0 = \sigma \tag{2.36a}$$

hence

$$\begin{aligned} \text{Re}\,\sigma &= \text{Im}\,(\omega\epsilon) \\ \text{Im}\,\sigma &= -\text{Re}\,(\omega\epsilon) \end{aligned} \tag{2.36b}$$

If we include the vacuum dielectric constant in ϵ, equation 2.36a can alternatively be written as

$$\omega_i = -\frac{\text{Re}\,\sigma}{[\partial(\omega\,\text{Re}\,\epsilon)]/\partial\omega} \tag{2.35b}$$

From equation 2.36b, one can conclude immediately that instability results either when a negative dissipation ($\text{Re}\,\sigma < 0$) couples to a positive-energy wave ($\partial(\omega\epsilon)/\partial\omega > 0$), such as the example in case 1, or when a negative energy wave ($\partial(\omega\epsilon)/\partial\omega < 0$) couples to a positive dissipation ($\text{Re}\,\sigma > 0$), as in case 2.

One can generalize the argument to a general dispersion relation (without restriction to an electrostatic mode) given by

$$D(\omega, \mathbf{k}) = 0 \tag{2.37}$$

if the following condition is satisfied by D: there exists real $\omega(=\omega_r)$ for a real $\mathbf{k}$ that satisfies

1. $$\text{Re}\,[D(\omega_r, \mathbf{k})] = 0 \tag{2.38}$$

2. $$|\text{Im}\,[D(\omega_r, \mathbf{k})]| \ll \left|\omega_r \frac{\partial\,\text{Re}\,[D(\omega_r, \mathbf{k})]}{\partial\omega_r}\right| \tag{2.39}$$

Using the same technique as before, by expanding D around $\omega = \omega_r$, we have

$$\omega_i = -\frac{\text{Im}\,D(\omega_r, \mathbf{k})}{\partial\,\text{Re}\,[D(\omega_r, \mathbf{k})]/\partial\omega} \tag{2.40}$$

In the magnetosphere, because there are different groups of plasmas (in terms of average energy) a situation often occurs when a wave propagated by one group (typically the cold- or low-energy group) resonates with particles in other groups (low-energy or medium- to high-energy group) and exchanges energy between groups. Under these circumstances, the condition of wave propagation (equation 2.33 or 2.38) is satisfied by the former group, whereas $\text{Re}\,\sigma$ in equation 2.35a or $\text{Im}\,D$ in equation 2.40 is decided by the latter group. Such an approach simplifies the analysis significantly. Therefore expressions (2.35a) or (2.40) are quite useful in magnetospheric plasmas.

Given the dispersion relation in the form of equation 2.37, finding a linear plasma instability requires only the algebra of finding a root for ω with a positive imaginary part for a given value of real wave number k. The above argument applies only for a case of a small growth rate. For a case with larger growth rate, a classic technique called Nyquist's theorem is useful. Consider the following Cauchy integral, I

$$I = \int_{\omega} \frac{d\omega}{D(\omega, k)} \times \frac{dD\,(\omega, k)}{d\omega} \tag{2.41}$$

where the integration contour is along the border of the upper half-plane in the complex ω plane, namely, from $-\infty$ to $+\infty$ on real ω axis and infinite semicircle on the upper half-plane (Figure 3). If $D = 0$ occurs for ω with a positive imaginary part, I has a finite value according to the Cauchy residue theorem. (D is assumed to have no pole. The assumption is shown to be valid for a physical system using causality arguments). Now, the integral I can be transformed into a D plane integral such as

$$I = \int_{\omega} \frac{dD/d\omega}{D}\,d\omega = \int_{D} \frac{dD}{D} \tag{2.42}$$

where the integration contour has to be mapped into the D plane. In the D plane, the pole occurs at $D = 0$, hence I has a value when the mapped contour in D plane encircles the origin in D plane. In other words, an instability results when the mapping on the D plane of the contour that encircles the upper half ω plane encircles the origin in the D plane. This is the Nyquist theorem (cf. Figure 3).

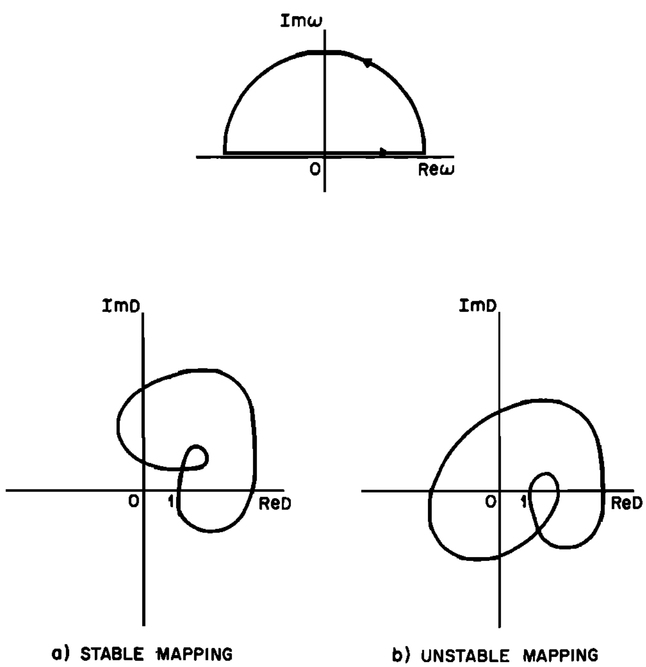

Fig. 3. Nyquist diagram for (a) stable and (b) unstable cases.

3. PLASMA INSTABILITIES RELEVANT IN THE MAGNETOSPHERE

In this chapter we summarize plasma instabilities that are possibly applicable to magnetospheric plasmas. A plasma instability occurs by conversion of steady free energy in the plasma into fluctuating field energy. Free energy is present when the plasma is not in thermodynamical equilibrium, that is, when the plasma is not either uniformly spread in space or does not have a Boltzmann distribution in velocity space. (Note, however, that a dynamical equilibrium state can be achieved if the forces acting on a plasma are balanced, even though the plasma is not in thermodynamical equilibrium.) Because the plasma in the magnetosphere fulfills neither criterion, it is subject to some instabilities.

3.1. Velocity-Space Instabilities

Plasma instabilities that originate from the velocity-space nonequilibrium are primarily due either to a two-humped velocity distribution or to an anisotropic distribution with respect to the ambient magnetic field or the direction of the wave propagation. Both electrostatic ($\nabla \times \mathbf{E} = 0$) and electromagnetic ($\nabla \cdot \mathbf{E} = 0$) modes become unstable in the presence of such distributions. For introductory purposes, we will start with one of the simplest examples of such an instability, an electrostatic instability caused by a two-humped distribution, and will gradually generalize the idea into more complicated systems.

3.1a. Electrostatic instabilities due to two-humped velocity distributions. If the velocity distribution function has two or more peaks, it is known that an instability can occur. The two-stream instability shown in chapter 2 is an example of this case. Whereas the two-stream instability discussed in chapter 2 is excited by the negative energy wave carried by the stream, in this subsection we discuss a two-stream instability that is excited by negative dissipation produced by wave-particle interactions.

Let us consider first an electrostatic perturbation in a high-frequency regime where the ion dynamics are negligible. When the velocity spread is significantly large, individual particles will have different trajectories, and some will interact with the wave strongly. On the other hand, the wave is produced by the macroscopic behavior of those particles, such as the number density or the current. Description of such a system is more effectively made by considering particle dynamics in phase space ($\mathbf{x}$, $\mathbf{v}$, t). When the collision effect is negligible, the phase-space density function $f(\mathbf{x}, \mathbf{v}, t)$, which represents the probability density in velocity space, of an individual particle can be regarded as the same on average as that of all the other particles, and its value is conserved.

The Vlasov equation, which represents the conservation of the density function $f(\mathbf{x}, \mathbf{v}, t)$ in a force field $\mathbf{F}(\mathbf{x}, t)$, can be written as (taking the nonrelativistic case for simplicity)

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{x}} + \frac{1}{m} \mathbf{F} \cdot \frac{\partial f}{\partial \mathbf{v}} = 0 \tag{3.1}$$

The current density $\mathbf{J}$ and the charge density ρ are obtained by taking the first and zeroth moments of f as

$$\mathbf{J}(\mathbf{x}, t) = qn_0 \int_{-\infty}^{\infty} \mathbf{v}f(\mathbf{x}, \mathbf{v}, t)\, d\mathbf{v} \tag{3.2}$$

$$\rho(\mathbf{x}, t) = qn_0 \int_{-\infty}^{\infty} f(\mathbf{x}, \mathbf{v}, t)\, d\mathbf{v} \tag{3.3}$$

where q is the charge of the particle represented by f, n_0 is the mean density, and $d\mathbf{v}$ represents the volume integral in the velocity space.

We first consider a case in which the direction of propagation is parallel to the magnetic field. Because we assume an electrostatic perturbation ($\nabla \times \mathbf{E} = 0$), we use the electrostatic potential φ to represent the field. The linearized Vlasov equation for the electron distribution function then becomes

$$\frac{\partial f_1}{\partial t} + v\frac{\partial f_1}{\partial z} + \frac{e}{m_e}\frac{\partial \varphi_1}{\partial z}\frac{\partial f_0}{\partial v} = 0 \tag{3.4}$$

and

$$F_1 = e(\partial\varphi_1/\partial z) \tag{3.5}$$

where f_1 [$= f_1(z, v, t)$] is the perturbed velocity distribution function of an electron, and f_0 [$= f_0(v)$] is the unperturbed velocity distribution function that depends only on velocity v. Substituting equation 3.5 into 3.4 and taking the Fourier-Laplace transformation as defined in equation 2.7, we have

$$f_1 = -\frac{(e/m_e)(\partial f_0/\partial v)}{v - \omega/k}\varphi_1 \tag{3.6}$$

The perturbed number density n_1 can be obtained then from equation 3.3 as

$$qn_1 = -en_0 \int_{-\infty}^{\infty} f_1\, dv = \frac{e^2 n_0}{m_e}\int_{-\infty}^{\infty} \frac{(\partial f_0/\partial v)\varphi_1}{v - \omega/k}\, dv \tag{3.7}$$

As was discussed in chapter 2, the Fourier-Laplace transformation is valid for the plane Im $\omega > 0$, hence, the integral over velocity space in equation 3.7 has to go below the pole at ω/k (see Figure 4). The conductivity associated with such an electron perturbation can then be obtained by using equation 2.30 as

$$\sigma = \frac{i\omega\epsilon_0\omega_{pe}^2}{k^2}\int_{-\infty}^{\infty} \frac{\partial f_0/\partial v}{v - (\omega/k)}\, dv \tag{3.8}$$

The dispersion relation is obtained by substituting the above expression into equation 2.32 as

$$1 - \frac{\omega_{pe}^2}{k^2}\int_{-\infty}^{\infty} \frac{\partial f_0/\partial v}{v - (\omega/k)}\, dv = 0 \tag{3.9}$$

According to the criterion presented in equation 2.23, the necessary condition of instability is obtained from the condition Re $\sigma < 0$. If the growth rate is very small, i.e., if Im ω is small, σ in equation 3.8 can be expressed in terms of the principal integral and the contribution from the pole at $v = \omega/k$ as (cf. Figure 4)

$$\sigma = \frac{i\omega\epsilon_0\omega_{pe}^2}{k^2}\left[P\int_{-\infty}^{\infty} \frac{\partial f_0/\partial v}{v - (\omega/k)}\, dv + i\pi \left.\frac{\partial f_0}{\partial v}\right|_{v=\omega/k}\right] \tag{3.10}$$

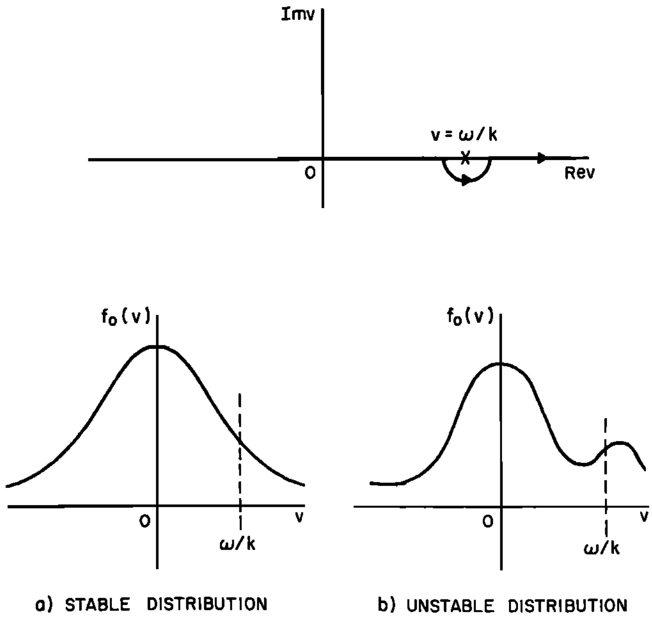

Fig. 4. Path of integration of equation 3.9 over velocity v, and the corresponding (a) stable and (b) unstable velocity distributions.

where P represents the principal integral. The necessary condition for instability with a small growth rate is then expressed immediately by

$$\omega \left. \frac{\partial f_0}{\partial v} \right|_{v=\omega/k} > 0 \tag{3.11}$$

Physically, equation 3.11 means that at $v = \omega/k(=v_p)$, the unperturbed velocity distribution function has a positive gradient. Since the Maxwell distribution of the form e^{-v^2} always has a negative gradient, it is always stable (Im $\omega < 0$, Landau damping); if the distribution has an additional peak at $v > \omega/k$, the necessary condition of the instability is satisfied. Given a two-humped distribution, the existence of a phase velocity in the range of positive gradient can be found only by solving for real ω and k the dispersion relation given by

$$1 - \frac{\omega_p^2}{k^2} P \int_{-\infty}^{\infty} \frac{\partial f_0/\partial v}{v - (\omega/k)}\, dv = 0 \tag{3.12}$$

For further details see *Landau* [1946] and *Jackson* [1960]. If it happens that equation 3.12 gives a complex ω solution, the above argument breaks down. For example, if f_0 represents two streams with no velocity spread, i.e., $f_0 = \delta(v) + \delta(v - v_0)$, equation 3.12 gives the same dispersion relation shown in the example in chapter 2, and has a complex ω solution. The instability occurs by interactions between negative and positive energy waves, and not by wave-particle interactions, as shown in equation 3.11.

We now consider a low frequency regime where ion dynamics become important. The dispersion relation including ion dynamics can be obtained in exactly the same manner by adding the ion charge density perturbation to the conductivity as

$$1 - \frac{\omega_{pe}^2}{k^2} \int_{-\infty}^{\infty} \frac{\partial f_{0e}/\partial v}{v - (\omega/k)} dv - \frac{\omega_{pi}^2}{k^2} \int_{-\infty}^{\infty} \frac{\partial f_{0i}/\partial v}{v - (\omega/k)} dv = 0 \tag{3.13}$$

where the ion and electron contributions are designated by the subscripts i and e. The dispersion relation given above is not analytically soluble for a general case. Hence we consider a particular case where the ions are cold and stationary, whereas the electrons are hot with their thermal velocity much larger than the phase velocity and drift with a drift velocity v_0 with respect to the ions. Then the integral for the ion contribution in equation 3.13 becomes, after integration by parts

$$I_i \equiv \int_{-\infty}^{\infty} \frac{\partial f_{0i}/\partial v}{v - (\omega/k)} dv = \int_{-\infty}^{\infty} \frac{f_{0i}}{[v - (\omega/k)]^2} dv$$

The 'cold' distribution can be represented by the delta function as

$$f_{0i} = \delta(v)$$

hence

$$I_i = k^2/\omega^2 \tag{3.14}$$

For electrons

$$\begin{aligned} I_e &= \int_{-\infty}^{\infty} \frac{\partial f_{0e}/\partial v}{v - (\omega/k)} dv \\ &= P \int_{-\infty}^{\infty} \frac{\partial f_{0e}/\partial v}{v - (\omega/k)} dv + i\pi \left.\frac{\partial f_{0e}}{\partial v}\right|_{v=\omega/k} \end{aligned}$$

because the thermal velocity for electrons is assumed much larger than the phase velocity, if we assume a Maxwellian distribution for f_{0e} such that

$$f_{0e} = \frac{1}{(2\pi)^{1/2} v_{Te}} e^{-(v-v_0)^2/2v_{Te}^2}$$

$$\begin{aligned} I_e &\cong \frac{1}{(2\pi)^{1/2} v_{Te}} \int_{-\infty}^{\infty} \frac{1}{v} \cdot \left(-\frac{v}{v_{Te}^2} e^{-v^2/2v_{Te}^2} \right) dv + i\pi \left.\frac{\partial f_{0e}}{\partial v}\right|_{v=\omega/k} \\ &= -\frac{1}{v_{Te}^2} + i\pi \left.\frac{\partial f_{0e}}{\partial v}\right|_{v=\omega/k} \end{aligned} \tag{3.15a}$$

where we assume the electron drift velocity v_0 to be much smaller than the thermal velocity v_{Te}. Then the dispersion relation equation 3.13 becomes

$$1 - \frac{\omega_{pi}^2}{\omega^2} + \frac{\omega_{pe}^2}{k^2 v_{Te}^2} - i\pi \frac{\omega_{pe}^2}{k^2} \left.\frac{\partial f_{0e}}{\partial v}\right|_{v=\omega/k} = 0 \tag{3.15b}$$

For a low frequency limit, $\omega \ll \omega_{pi}$, the real part of the dispersion relation is readily soluble and gives

$$\omega = kc_s \tag{3.16}$$

where c_s is the ion sound speed given by

$$c_s = v_{Te}(m_e/m_i)^{1/2} \tag{3.17}$$

The wave represented by the dispersion relation (3.16) is called the ion acoustic wave. The growth rate γ of the instability is obtained using equation 2.34a as

$$\gamma = \pi\omega v_{Te}^{2} \left.\frac{\partial f_{0e}}{\partial v}\right|_{v=c_s} \tag{3.18}$$

As expected, the instability ($\gamma > 0$) occurs when $\partial f_{0e}/\partial v$ is positive at $v = c_s$. This means the drift velocity v_0 of the electrons has to be greater than c_s because f_{0e} has its only peak at $v = v_0$.

We have shown that when the ions are cold and the electrons are hot the ion sound wave becomes unstable when the electron drift velocity exceeds the ion sound speed. However, if the ion temperature is comparable with the electron temperature, the instability becomes possible only when the electron drift speed exceeds the electron thermal speed, v_{Te}, which is 43 $[=(m_i/m_e)^{1/2}]$ times larger than c_s [*Fried and Gould*, 1961].

3.1b. Electrostatic instabilities due to anisotropic velocity distributions in the presence of a uniform magnetic field. In this subsection, we consider the effect of an ambient uniform magnetic field on electrostatic waves. The Vlasov equation for the unperturbed velocity distribution function f_0 becomes

$$(\mathbf{v} \times \mathbf{B}_0)\cdot\frac{\partial f_0}{\partial \mathbf{v}} = 0 \tag{3.19}$$

Equation 3.19 can be satisfied for any function of $v_\perp$ and $v_\parallel$ as

$$f_0(v) = f_0(v_\perp, v_\parallel) \tag{3.20a}$$

where

$$v_\parallel = v_z$$

and (3.20b)

$$v_\perp = (v_x^2 + v_y^2)^{1/2}$$

are the velocities parallel and perpendicular to the magnetic field, and z is taken to be parallel to $\mathbf{B}_0$. As was pointed out before, the thermodynamic equilibrium situation is achieved when f_0 is the isotropic Boltzmann distribution; hence, any anisotropic distribution or two-humped distribution either in $v_\perp$ or $v_\parallel$ produces free energy in the plasma and may cause instabilities.

If we limit our interest still to electrostatic perturbations, the linearized Vlasov equation for a species with charge q and mass m becomes

$$\frac{\partial f_1}{\partial t} + \mathbf{v}\cdot\frac{\partial f_1}{\partial \mathbf{x}} + \frac{q}{m}(\mathbf{v} \times \mathbf{B}_0)\cdot\frac{\partial f_1}{\partial \mathbf{v}} = \frac{q}{m}\frac{\partial \varphi_1}{\partial \mathbf{x}}\cdot\frac{\partial f_0}{\partial \mathbf{v}} \tag{3.21}$$

In equation 3.21 f_1 can be obtained either by integrating along the unperturbed orbit [for example, see *Krall*, 1968] or by solving the differential equation by a suitable change of variables [for example, see *Bernstein*, 1958]. We take the former method because it has more general application, as will be seen later. f_1 then can be written formally

$$f_1(\mathbf{x}, \mathbf{v}, t) = \int_{-\infty}^{t} dt' \frac{q}{m} \frac{\partial \varphi_1(t')}{\partial \mathbf{x}'} \cdot \frac{\partial f_0}{\partial \mathbf{v}'} \tag{3.22}$$

where the unperturbed orbit of the particle can be obtained from the equation of motion in the unperturbed field

$$d\mathbf{v}'/dt' = (q/m)(\mathbf{v}' \times \mathbf{B}_0) \tag{3.23}$$

which gives

$$\begin{aligned} x' &= x_0 + \frac{v_\perp'}{\omega_c} [\cos(\theta - \omega_c t') - \cos\theta] \\ y' &= y_0 + \frac{v_\perp'}{\omega_c} [\sin(\theta - \omega_c t') - \sin\theta] \\ z' &= z_0 + v_\parallel' t' \end{aligned} \tag{3.24}$$

where θ is the polar angle in velocity space and $\omega_c(=qB_0/m)$ is the cyclotron frequency.

If we consider a perturbation of the form $\exp i(\mathbf{k}\cdot\mathbf{x} - \omega t)$ and take the direction of wave propagation in the x, z plane (without loss of generality) such that

$$\mathbf{k}\cdot\mathbf{x} = k_\perp x + k_\parallel z \tag{3.25}$$

Equation 3.22 gives

$$f_1(\mathbf{x}, \mathbf{v}, t) = i \int_{-\infty}^{t} dt' \left(k_\perp \frac{\partial f_0}{\partial v_x'} + k_\parallel \frac{\partial f_0}{\partial v_\parallel'}\right) \varphi_1 \exp i[k_\perp x'(t') + k_\parallel z'(t') - \omega t'] \tag{3.26}$$

After performing the integral over t' and substituting the result into equation 3.4 to obtain the number-density perturbation n_1, one can obtain the longitudinal conductivity of the plasma in a magnetic field from equation 2.30 as

$$\sigma(\omega, k) = \frac{i\omega\epsilon_0\omega_p^2}{k^2} \sum_{n=-\infty}^{\infty} \int d\mathbf{v} \left[J_n\left(\frac{k_\perp v_\perp}{\omega_c}\right)\right]^2 \frac{k_\parallel(\partial f_0/\partial v_\parallel) + (n\omega_c/v_\perp)(\partial f_0/\partial v_\perp)}{k_\parallel v_\parallel - (\omega - n\omega_c)} \tag{3.27}$$

where $d\mathbf{v} = 2\pi v_\perp\, dv_\perp\, dv_\parallel$ and J_n is the nth order Bessel function of the first kind. In deriving equation 3.27 use is made of the identity

$$\exp(iz\sin\theta) = \sum_{n=-\infty}^{\infty} J_n(z) \exp(in\theta).$$

Because we already know that a two-humped distribution in $v_\parallel$ produces a conductivity with a negative dissipation, we assume a stable (single-humped) distribution in the parallel direction. We take a Maxwellian distribution for $v_\parallel$; then

$$f_0(v_\parallel, v_\perp) = \frac{1}{(2\pi)^{1/2}} \frac{1}{v_{T\parallel}} \exp\left(-\frac{v_\parallel^2}{2v_{T\parallel}^2}\right) f_{0\perp}(v_\perp) \tag{3.28}$$

where $v_{T\parallel}$ is the thermal velocity in the parallel direction and $f_{0\perp}$ is the distribution function depending only on the perpendicular velocity $v_\perp$. It is also convenient to introduce here the plasma dispersion function defined and tabulated by *Fried and Conte* [1961]

$$Z(\zeta) = \frac{1}{(\pi)^{1/2}} \int_{-\infty}^{\infty} \frac{e^{-x^2}}{x - \zeta} dx \quad \text{for} \quad \text{Im } \zeta > 0 \tag{3.29}$$

$$= \text{analytic continuation of the above integral for Im } \zeta < 0$$

The power-series expansion for a small argument and the asymptotic expansion for a large argument of the Z function are given by

$$Z(\zeta) \cong i(\pi)^{1/2} e^{-\zeta^2} - 2\zeta\left(1 - \frac{2\zeta^2}{3}\right) \quad \text{for} \quad |\zeta| \ll 1 \tag{3.30}$$

$$Z(\zeta) \cong i(\pi)^{1/2} \sigma e^{-\zeta^2} - \frac{1}{\zeta}\left(1 + \frac{1}{2\zeta^2}\right) \quad \text{for} \quad |\zeta| \gg 1 \tag{3.31}$$

where

$$\sigma = \begin{cases} 0 & \text{Im } \zeta > 0 \\ 1 & \text{Im } \zeta = 0 \\ 2 & \text{Im } \zeta < 0 \end{cases}$$

Then equation 3.24 may be expressed as

$$\sigma = -\frac{i\omega\epsilon_0\omega_p^2}{k^2 v_{T\parallel}^2} \sum_{n=-\infty}^{\infty} F_n\left[1 + \frac{\omega - n\omega_c(1 - R_n v_{T\parallel}^2/v_{T\perp}^2)}{(2)^{1/2} k_\parallel v_{T\parallel}} Z\left(\frac{\omega - n\omega_c}{(2)^{1/2} k_\parallel v_{T\parallel}}\right)\right] \tag{3.32}$$

where

$$F_n = \int_0^\infty \left[J_n\left(\frac{k_\perp v_\perp}{\omega_c}\right)\right]^2 f_{0\perp}(v_\perp) 2\pi v_\perp \, dv_\perp > 0 \tag{3.33}$$

and

$$R_n = -\frac{v_{T\perp}^2}{F_n} \int_0^\infty \left[J_n\left(\frac{k_\perp v_\perp}{\omega_c}\right)\right]^2 \frac{1}{v_\perp} \frac{\partial f_{0\perp}}{\partial v_\perp} 2\pi v_\perp \, dv_\perp \tag{3.34}$$

where

$$v_{T\parallel}^2 = \langle v_\parallel^2 \rangle \quad \text{and} \quad v_{T\perp}^2 = \langle v_\perp^2 \rangle / 2$$

The necessary condition of the instability is again obtained by finding the condition Re $\sigma < 0$. Because the imaginary part of the plasma dispersion function is always positive, such a condition can be written as

$$0 < \omega < n\omega_c(1 - v_{T\parallel}^2 R_n / v_{T\perp}^2) \tag{3.35a}$$

or

$$1 - v_{T\parallel}^2 R_n / v_{T\perp}^2 > 0 \tag{3.35b}$$

Now R_n is shown to be unity if $f_{0\perp}$ is also Maxwellian (but with perpendicular thermal velocity $v_{T\perp}$ different from the parallel thermal velocity $v_{T\parallel}$). Then by writing $v_{T\parallel}^2/v_{T\perp}^2 = T_\parallel/T_\perp$, where $T_\parallel$ and $T_\perp$ are parallel and perpendicular temperatures, respectively, (3.35*b*) reduces to

$$T_\perp/T_\parallel > 1 \qquad (3.35c)$$

Equation 3.35*c* shows that a plasma with a larger temperature in the direction perpendicular to the magnetic field than that in the parallel direction can become unstable for an electrostatic perturbation.

If $f_{0\perp}$ is not Maxwellian, R_n can deviate from unity. In fact if $f_{0\perp}$ has a peak at $v_\perp \neq 0$, R_n can become negative, in which case the necessary condition of instability (3.35*b*) is satisfied independent of the ratio $v_{T\parallel}/v_{T\perp}$. Hence another necessary condition, which is independent of equation 3.35*c*, is

$$\int_0^\infty \left[J_n\left(\frac{k_\perp v_\perp}{\omega_c}\right)\right]^2 \frac{1}{v_\perp}\frac{\partial f_{0\perp}}{\partial v_\perp} 2\pi v_\perp \, dv_\perp > 0 \qquad (3.35d)$$

Equations 3.35*a*, *b*, *c*, and *d* are necessary conditions for an instability of electrostatic perturbations in a magnetic field. The necessary and sufficient condition must be worked out by finding a suitable wave mode with which the negative dissipation found here can couple. These instabilities, which are associated with anisotropies in the velocity distribution, were found by *Harris* [1961].

In addition to the distribution anisotropy, low-energy particles in the magnetospheric plasma tend to have a distribution in which particles falling into the loss cone are absent. Such a distribution, called a loss-cone distribution, is known also to be subject to an electrostatic instability [*Post and Rosenbluth*, 1966].

3.1c. Electromagnetic instabilities due to anisotropic velocity distributions in the presence of a uniform magnetic field. In this subsection we consider plasma instabilities that have a predominantly electromagnetic nature. This means the case where $\nabla \cdot \mathbf{E} \sim 0$ in contrast to the electrostatic cases where $\nabla \times \mathbf{E} \sim 0$. In reality, however, waves in a hot plasma propagating oblique to the ambient magnetic field satisfy neither $\nabla \times \mathbf{E} = 0$ nor $\nabla \cdot \mathbf{E} = 0$. Only when the $\mathbf{k}$ vector is exactly parallel or perpendicular to the magnetic field do the electrostatic waves and electromagnetic waves separate.

In the case of oblique propagation ($\mathbf{k} \times \mathbf{B}_0 \neq 0$), however, one can generally say that the wave approaches electrostatic near the resonant points given by $|\mathbf{k}| \to \infty$. Hence the results presented in subsection 3.1*b* are valid for short wavelength. In the general case of oblique propagation, however, one has to derive the dispersion relation from the full Maxwellian equations

$$\mathbf{k}(\mathbf{k}\cdot\mathbf{E}_1) - k^2\mathbf{E}_1 + \frac{\omega^2}{c^2}(\mathsf{I} + \boldsymbol{\varepsilon})\cdot\mathbf{E}_1 = 0 \qquad (3.36)$$

where I is a unit tensor given by

$$\mathsf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (3.37)$$

and $\boldsymbol{\varepsilon}$ is the equivalent dielectric tensor that is related to the conductivity tensor by

$$\boldsymbol{\varepsilon} = -\frac{\boldsymbol{\sigma}}{i\omega\epsilon_0} \tag{3.38}$$

and

$$\begin{aligned} \mathbf{J}_1 &\equiv qn_0 \int \mathbf{v} f_1 \, d\mathbf{v} \\ &\equiv \boldsymbol{\sigma} \cdot \mathbf{E}_1 \end{aligned} \tag{3.39}$$

The perturbed distribution function f_1 can be obtained as a function of $\mathbf{E}_1$ by solving the Vlasov equation including the full electromagnetic field

$$\frac{\partial f_1}{\partial t} + \mathbf{v} \cdot \frac{\partial f_1}{\partial \mathbf{x}} + \frac{q}{m} (\mathbf{v} \times \mathbf{B}_0) \cdot \frac{\partial f_1}{\partial \mathbf{v}} = -\frac{q}{m} \left[\mathbf{E}_1 + \frac{\mathbf{v} \times (\mathbf{k} \times \mathbf{E}_1)}{\omega} \right] \cdot \frac{\partial f_0}{\partial \mathbf{v}} \tag{3.40}$$

The equivalent dielectric tensor obtained this way has the following expression

$$\boldsymbol{\varepsilon} = -\sum_{\text{species}} \frac{\omega_p^2}{\omega^2} \left\{ \mathsf{I} + \sum_{n=-\infty}^{\infty} \int d\mathbf{v} \, \frac{k_\parallel (\partial f_0 / \partial v_\parallel) + (n\omega_c / v_\perp)(\partial f_0 / \partial v_\perp)}{k_\parallel v_\parallel - (\omega - n\omega_c)} \times \mathsf{S} \right\} \tag{3.41}$$

where the matrix I is given by equation 3.37 and the matrix S is given by

$$\mathsf{S} = \begin{bmatrix} \left(\frac{n\omega_c}{k_\perp} J_n \right)^2 & i \frac{n\omega_c}{k_\perp} v_\perp J_n J_n' & \frac{n\omega_c}{k_\perp} v_\parallel J_n^2 \\ -i \frac{n\omega_c}{k_\perp} v_\perp J_n J_n' & (v_\perp J_n')^2 & -i v_\perp v_\parallel J_n J_n' \\ \frac{n\omega_c}{k_\perp} v_\parallel J_n^2 & i v_\perp v_\parallel J_n J_n' & (v_\parallel J_n)^2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{3.42}$$

and the arguments of the Bessel functions J_n are $k_\perp v_\perp / \omega_c$ and ω_c is the cyclotron frequency qB_0/m with sign included. In equation 3.42, $\mathbf{B}_0$ is taken in the direction of positive z axis. Expressions 3.41 and 3.42 are those derived by *Guest* [1971]. The dispersion relation obtainable from equations 3.36 and 3.41 above represents the most general case for a uniform plasma in a uniform magnetic field. For example, instabilities presented in subsection 3.1*b* can be derived in a more exact form using the above expression. We do not, however, go into the detail of electromagnetic modification of the instabilities derived in subsection 3.1*b*, but focus our attention only on instabilities that are predominantly electromagnetic. For this purpose we look at the case where the wave propagates parallel to the magnetic field. In this case, instability arises owing to either two-streaming or pitch-angle anisotropy $(T_\perp / T_\parallel > 1)$. Waves associated with these instabilities are the electron- and proton-cyclotron waves.

In Maxwell's equation 3.36 we put $\mathbf{k} \cdot \mathbf{E}_1 = 0$. If we also put $k_\perp = 0$ in the dielectric tensor in equation 3.41, ϵ_{yy} becomes equal to ϵ_{xx}, and ϵ_{yz} as well as ϵ_{zz} vanishes. We then have for a transverse wave propagating parallel to the magnetic field

$$\begin{aligned} -k^2 E_x + \frac{\omega^2}{c^2} [(1 + \epsilon_{xx})(E_x + \epsilon_{xy} E_y)] &= 0 \\ -k^2 E_y + \frac{\omega^2}{c^2} [(1 + \epsilon_{xx})(E_y - \epsilon_{xy} E_x)] &= 0 \end{aligned} \tag{3.43}$$

If we define a new electric-field vector

$$E_R = E_x - iE_y \tag{3.44}$$

for the right-hand circularly polarized wave and

$$E_L = E_x + iE_y \tag{3.45}$$

for the left-hand circularly polarized wave, E_R and E_L satisfy

$$\left[-k^2 + \frac{\omega^2}{c^2}(1 + \epsilon_{xx}) + i\frac{\omega^2}{c^2}\epsilon_{xy}\right]E_R = 0 \tag{3.46}$$

and

$$\left[-k^2 + \frac{\omega^2}{c^2}(1 + \epsilon_{xx}) - i\frac{\omega^2}{c^2}\epsilon_{xy}\right]E_L = 0 \tag{3.47}$$

From equations 3.46 and 3.47, together with the dielectric tensor in equation 3.41, we have the dispersion relation for the right-hand polarized wave

$$k^2c^2 - \omega^2 + \omega_{pe}^2 \int \frac{(\omega - kv_\parallel)f_{0\parallel}^e(v_\parallel) - k[(\langle v_{\perp e}^2\rangle/2)(\partial f_{0\parallel}^e/\partial v_\parallel)]}{\omega - kv_\parallel - \omega_{ce}}\,dv_\parallel$$
$$+ \omega_{pi}^2 \int \frac{(\omega - kv_\parallel)f_{0\parallel}^i(v_\parallel) - k[(\langle v_{\perp i}^2\rangle/2)(\partial f_{0\parallel}^i/\partial v_\parallel)]}{\omega - kv_\parallel + \omega_{ci}}\,dv_\parallel = 0 \tag{3.48}$$

and for the left-hand polarized wave

$$k^2c^2 - \omega^2 + \omega_{pe}^2 \int \frac{(\omega - kv_\parallel)f_{0\parallel}^e(v_\parallel) - k[(\langle v_{\perp e}^2\rangle/2)(\partial f_{0\parallel}^e/\partial v_\parallel)]}{\omega - kv_\parallel + \omega_{ce}}\,dv_\parallel$$
$$+ \omega_{pi}^2 \int \frac{(\omega - kv_\parallel)f_{0\parallel}^i(v_\parallel) - k[(\langle v_{\perp i}^2\rangle/2)(\partial f_{0\parallel}^i/\partial v_\parallel)]}{\omega - kv_\parallel - \omega_{ci}} = 0 \tag{3.49}$$

In the above expressions, $f_{0\parallel}^e$ and $f_{0\parallel}^i$ are the electron and ion distribution functions in the parallel direction, and $\langle v_{\perp e}^2\rangle/2$ and $\langle v_{\perp i}^2\rangle/2$ are the electron and ion thermal velocities in the perpendicular direction. In equations 3.48 and 3.49, if we take a limit of zero temperature by putting $\langle v_\perp^2\rangle = 0$ and $f_{0\parallel} = \delta(v_\parallel)$, we can recover the familiar dispersion relation for electron-cyclotron and ion cyclotron waves, respectively.

Let us first discuss the two-streaming-type instability. For this we choose a coordinate system fixed to the ions such that the distribution functions are given by

$$\begin{aligned} f_{0\parallel}^e &= \delta(v_\parallel - v_0) \\ f_{0\parallel}^i &= \delta(v_\parallel) \end{aligned} \tag{3.50}$$

Substituting equation 3.50 into 3.49 we obtain the dispersion relation of ion-cyclotron waves with drifting cold electrons

$$k^2c^2 - \omega^2 + \omega_{pe}^2 \frac{\omega - kv_0}{\omega - kv_0 + \omega_{ce}} + \omega_{ci}^2 \frac{\omega}{\omega - \omega_{ci}} = 0 \tag{3.51}$$

The instability arising from equation 3.51 was first discussed by *Bernstein and Dawson* [1958]. However, *Briggs* [1964] as well as *Hasegawa and Birdsall* [1964] later pointed out that there are two ranges in ω ($\omega \approx 0$ and $\omega \approx \omega_{ci}$) and

k ($k \approx 0$ and $k \approx \omega_{ce}/v_0$) in which such an instability arises, as can be seen from the plot of equation 3.51 in (ω, k) space in Figure 5. More precisely, the distribution function given by equation 3.50 produces current in the z direction; hence, the assumption of the uniform z-directed magnetic field is violated. One must therefore assume an ion current to compensate this electron current.

It is important to point out that, although for the example shown here the two-stream-type of cyclotron-wave instability occurs only for streams of opposite charge sign, a two-stream instability between the same species (electron-electron or ion-ion) can occur for cyclotron waves if one considers either the effect of finite cross section of the stream [*Hasegawa,* 1966] or the effect of anisotropic kinetic energy [*Bell and Buneman*, 1964].

We will now discuss cyclotron-wave instabilities associated with anisotropic temperature ($T_\perp > T_\parallel$). Such instabilities were first pointed out by *Rosenbluth* [1959] and *Weibel* [1959]. As in the case of the electrostatic two-stream instability, the instability arises because of either the negative-energy wave or of negative dissipation. The instability due to the negative-energy wave can be found easily by considering the limit of $T_\parallel \rightarrow 0$ with finite $T_\perp$. For example, the dispersion relation of the left-hand polarized wave then becomes

$$k^2\left[1 + \frac{\omega_{pe}^2\langle v_{\perp e}^2\rangle/2}{(\omega + \omega_{ce})^2c^2} + \frac{\omega_{pi}^2\langle v_{\perp i}^2\rangle/2}{(\omega - \omega_i)^2c^2}\right] = \frac{\omega^2}{c^2}\left[1 - \frac{\omega_{pe}^2}{\omega(\omega + \omega_{ce})} - \frac{\omega_{pi}^2}{\omega(\omega - \omega_{ci})}\right] \quad (3.52)$$

From equation 3.52, ω is found to become complex for wave number $k > k_m$, where

$$k_m^2 \sim \frac{\omega_{pi}\omega_{ci}}{c(\langle v_{\perp i}^2\rangle)^{1/2}} \quad (3.53)$$

and the growth rate for $k \gg k_m$ becomes

$$\operatorname{Im}\omega \approx \frac{\omega_{pi}}{c}\left(\frac{\langle v_{\perp i}^2\rangle}{2}\right)^{1/2} \quad (3.54)$$

while the real part of the frequency is given by

$$\operatorname{Re}\omega \approx \omega_c, \quad (3.55)$$

The dispersion relation equation 3.52 is plotted in Figure 6.

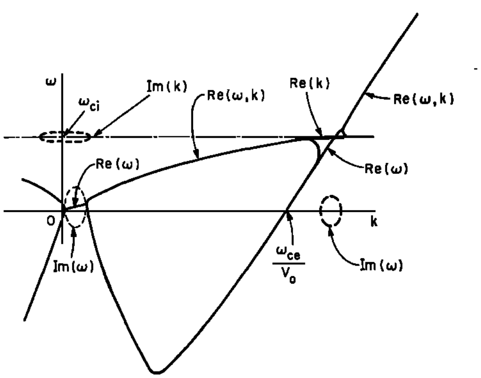

Fig. 5. Dispersion diagram of coupling between the ion-cyclotron wave and the slow electron-cyclotron wave in a two-stream electron-ion system.

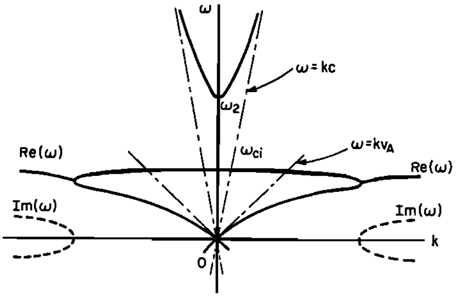

Fig. 6. Dispersion diagram of the ion-cyclotron wave in a plasma with anistropic temperature ($T_\perp \gg T_\parallel$).

When there exists a significant spread in $v_\parallel$ and a large number of particles resonate with the wave at $\omega = \omega_c \pm kv_\parallel$, Im ϵ becomes sizably large, and the instability becomes the negative-dissipative type. We take the electron cyclotron wave as an example. If the growth rate Im ω is small, we have from equation 3.48

$$\operatorname{Im}\omega \propto \operatorname{Im}\int \frac{(\omega - kv_\parallel)f_{0\parallel}{}^e(v_\parallel) - k[(\langle v_{\perp e}{}^2\rangle/2)(\partial f_{0\parallel}{}^e/\partial v_\parallel)]}{\omega - kv_\parallel - \omega_{ce}}\, dv_\parallel \tag{3.56}$$

For most distributions $\partial f_{0\parallel}{}^e/\partial v_\parallel$ may be written as $-v_\parallel/\langle v_{\parallel e}{}^2\rangle f_{0\parallel}{}^e$. Thus if we write

$$\frac{\langle v_{\perp e}{}^2\rangle}{2\langle v_{\parallel e}{}^2\rangle} \equiv \frac{T_{\perp e}}{T_{\parallel e}} \tag{3.57}$$

equation 3.56 becomes

$$\operatorname{Im}\omega \propto \int \frac{(\omega - kv_\parallel + kv_\parallel T_{\perp e}/T_{\parallel e})f_{0\parallel}{}^e}{\omega - kv_\parallel - \omega_{ce}}\, dv_\parallel$$

which can be further reduced for small Im ω using Dirac's expression as in equation 3.10

$$\operatorname{Im}\omega \sim -\frac{\pi}{k}\left[\omega_{ce} + (\omega - \omega_{ce})\frac{T_{\perp e}}{T_{\parallel e}}\right] f_{0\parallel}{}^e\left(\frac{\omega - \omega_{ce}}{k}\right) \tag{3.58}$$

The condition of the instability is obtained simply by imposing Im $\omega > 0$ and

$$\frac{T_{\perp e}}{T_{\parallel e}} > \frac{\omega_{ce}}{\omega_{ce} - \omega} \tag{3.59a}$$

Equation 3.59*a* shows that even for a small anisotropy in temperature, Im ω becomes positive for $\omega \to \omega_{ce}$. When Im ω is large, the necessary and sufficient condition of the instability can be obtained by finding the positive-energy range for the frequency range satisfying equation 3.59*a* (cf. equation 2.35*b*). Because we know from equation 3.54 that a cyclotron wave with anisotropic temperature carries a negative-energy wave for a frequency range similar to that for which Im $\omega > 0$, equation 3.59*a* does not give the necessary and sufficient condition of the instability in general.

However, as in the magnetosphere, if a significant amount of cold electrons are present, the cyclotron wave carried by the cold electrons is definitely a positive energy wave; hence, equation 3.59*a* will give the necessary and sufficient condition for the instability. This reasoning is applied by *Kennel and Petschek* [1966] in their calculation of pitch-angle scattering of electrons by the waves excited by this instability. Isotropization of the distribution as a consequence of the instability has been demonstrated by *Hasegawa and Birdsall* [1964] using a computer experiment.

This electromagnetic-wave instability excited by a temperature anisotropy exists even without a magnetic field as shown by *Weibel* [1959] when the temperature perpendicular to the direction of propagation is larger than the parallel temperature. As can be seen from expressions 3.48 and 3.49, a large $\langle v_\perp^2\rangle$ is needed for the instability to occur; hence, it is not necessarily a large 'temperature,' but can be a kinetic energy in the perpendicular direction. In the absence of the ambient magnetic field, a two stream in one species produces $\langle v_\perp^2\rangle$ and hence is subject to an instability. Such an instability, the electromagnetic-wave instability propagating in the direction perpendicular to the stream, has been discussed by *Momota* [1966].

3.1d. Hydromagnetic instabilities due to anisotropic pressure. In this subsection we discuss the effects of pressure anisotropy on hydromagnetic waves (low-frequency electromagnetic waves in a magnetoplasma). Hydromagnetic instabilities arising from anisotropic pressure can be derived from the combination of Maxwell's equation 3.36 and the equivalent dielectric tensor given in equations 3.41 and 3.42, by taking the limit as $\omega/\omega_{ci} \ll 1$ and $kv_{Ti}/\omega_{ci} \ll 1$. As was shown by *Kutsenko and Stepanov* [1960], at such low frequency and long wavelength limits the dispersion relation separates into two expressions:

$$(c^2k_\parallel^2/\omega^2) - \epsilon_{xx} = 0 \tag{3.60a}$$

representing the shear mode (the mode having no variation in the parallel component of the magnetic field) and

$$\frac{c^2k^2}{\omega^2} - \left(\epsilon_{yy} + \frac{\epsilon_{yz}^2}{\epsilon_{zz}}\right) = 0 \tag{3.61a}$$

which represents the coupled compressional wave and the ion acoustic wave (the mode having variation in the parallel component of the magnetic field). In the above expressions, ϵ's are the components of the dielectric tensor at the hydromagnetic limit, which can be expressed in general as [*Hasegawa*, 1970*a*]

$$\begin{aligned}
\epsilon_{xx} &= \sum_{\text{species}} \frac{\omega_p^2}{\omega_c^2}\left[\left\langle\left(1 - \frac{k_\parallel v_\parallel}{\omega}\right)^2\right\rangle - \frac{k_\parallel^2\langle v_\perp^2\rangle}{2\omega^2}\right] \\
\epsilon_{xy} &= -\epsilon_{yx} = i \sum_{\text{species}} \frac{\omega_p^2}{\omega\omega_c}\left(1 - \frac{k_\parallel\langle v_\parallel\rangle}{\omega}\right) \\
\epsilon_{yy} &= \epsilon_{xx} - \sum_{\text{species}} \frac{\omega_p^2}{\omega_c^2}\left(\frac{k_\perp^2\langle v_\perp^2\rangle}{\omega^2} + \frac{k^2\langle v_\perp^4\rangle}{4\omega^2} I\right) \\
\epsilon_{zz} &= -\sum_{\text{species}}\left[\frac{\omega_p^2}{k_\parallel^2} I + \frac{\omega_p^2}{\omega_c^2}\left(\frac{k_\perp^2\langle v_\perp^2\rangle}{2\omega^2} - \frac{k_\perp^2\langle v_\parallel^2\rangle}{\omega^2}\right)\right]
\end{aligned} \tag{3.62}$$

where

$$I = \int_{-\infty}^{\infty} dv_\parallel \int_0^{\infty} 2\pi v_\perp \, dv_\perp \frac{f_0(v_\parallel, v_\perp)}{(v_\parallel - \omega/k_\parallel)^2}$$

As was pointed out by the author [*Hasegawa,* 1969], in the magnetosphere where at least 10% of cold electron mixture is expected ϵ_{zz} becomes very large, and E_z is effectively short-circuited by the light cold electrons in most cases. In this case the ion acoustic mode decouples from the compressional Alfvén wave, and equation 3.61*a* further reduces to

$$\frac{c^2k^2}{\omega^2} - \epsilon_{yy} = 0 \qquad (3.61b)$$

Let us consider an instability associated with the shear Alfvén mode represented by equation 3.60*a*. For simplicity, if we consider a nondrifting plasma, $\langle v_\parallel \rangle = 0$, the dispersion relation becomes, with equation 3.62

$$\frac{c^2k_\parallel^2}{\omega^2} - \sum_{\text{species}} \frac{\omega_p^2}{\omega_c^2}\left[1 + \frac{k_\parallel^2}{\omega^2}\left(\langle v_\parallel^2 \rangle - \frac{\langle v_\perp^2 \rangle}{2}\right)\right] = 0 \qquad (3.60b)$$

If we note that

$$\frac{\omega_{pe}^2}{c^2\omega_{ce}^2} \ll \frac{\omega_{pi}^2}{c^2\omega_{ci}^2} = \frac{1}{v_A^2} \qquad (3.63)$$

where v_A is the Alfvén speed and

$$\frac{\omega_p^2\langle v^2 \rangle}{\omega_c^2c^2} = \frac{1}{2}\frac{mn\langle v^2 \rangle}{B_0^2/2\mu_0} = \frac{\beta}{2} \qquad (3.64)$$

for each species, where β is the pressure ratio of a plasma species to the magnetic field, then equation 3.60*b* reduces to

$$\frac{\omega^2}{k_\parallel^2 v_A^2} = 1 - \sum_{\text{species}} \tfrac{1}{2}(\beta_\parallel - \beta_\perp) \qquad (3.60c)$$

Equation 3.60*c*, which gives the standard dispersion relation for the shear Alfvén wave if $\beta_\parallel = \beta_\perp$, can be seen to produce imaginary ω if

$$1 - \sum_{\text{species}} \tfrac{1}{2}(\beta_\parallel - \beta_\perp) < 0 \qquad (3.65)$$

Equation 3.65 gives the necessary and sufficient condition of the instability of shear Alfvén waves. Instability occurs for $\beta_\parallel > \beta_\perp \sim 1$, namely for a high β plasma with a parallel plasma pressure larger than the perpendicular pressure. This instability is called the hose instability because of its similarity to the instability of a garden hose caused by a large parallel flow of water.

Let us now look at the instability associated with the compressional mode given by equation 3.61*b*. If we consider again a nondrifting plasma with a Maxwellian velocity distribution, ϵ_{yy} becomes, from equation 3.62

$$\epsilon_{yy} = \sum_{\text{species}} \left\{\frac{\omega_p^2}{\omega^2} - \frac{c^2}{\omega^2}\left[k_\parallel^2 \frac{\beta_\perp - \beta_\parallel}{2} + k_\perp^2\beta_\perp\left(1 + \frac{\beta_\perp}{2\beta_\parallel} Z'\left(\frac{\omega}{(2)^{1/2}k_\parallel v_\parallel}\right)\right)\right]\right\} \qquad (3.66)$$

where Z' is the derivative of the plasma dispersion function defined in equation 3.29 and given by

$$Z'(\zeta) = -2[1 + \zeta Z(\zeta)] \tag{3.67}$$

The unstable root is found by taking a low phase velocity limit such that

$$\omega/k_\parallel v_{\parallel i} \ll 1 \tag{3.68}$$

and

$$\omega^2/k^2 v_A{}^2 \ll 1 \tag{3.69}$$

in which case the dispersion relation of the compressional mode (3.61*b*) reduces to

$$k_\parallel{}^2\left(1 + \sum_{\text{species}} \frac{\beta_\perp - \beta_\parallel}{2}\right) + k_\perp{}^2\left[1 + \sum_{\text{species}} \beta_\perp\left(1 - \frac{\beta_\perp}{\beta_\parallel}\right) - i\,\frac{\beta_{\perp i}{}^2}{\beta_{\parallel i}}\,\frac{\omega}{k_\parallel\langle v_{\parallel i}\rangle}\left(\frac{\pi}{2}\right)^{1/2}\right] = 0 \tag{3.61c}$$

Solving for ω, we have

$$\omega = -ik_\parallel\langle v_{\parallel i}\rangle\left(\frac{2}{\pi}\right)^{1/2}\frac{\beta_{\parallel i}}{\beta_{\perp i}{}^2}\left[\frac{k_\parallel{}^2}{k_\perp{}^2}\left(1 + \sum_{\text{species}} \frac{\beta_\perp - \beta_\parallel}{2}\right) + 1 + \sum_{\text{species}} \beta_\perp\left(1 - \frac{\beta_\perp}{\beta_\parallel}\right)\right] \tag{3.70}$$

One can immediately see from this expression that the instability (Im $\omega > 0$) occurs either when

$$1 + \sum_{\text{species}} \frac{\beta_\perp - \beta_\parallel}{2} < 0 \tag{3.71}$$

for large $k_\parallel/k_\perp$ (almost parallel propagation), or when

$$1 + \sum_{\text{species}} \beta_\perp\left(1 - \frac{\beta_\perp}{\beta_\parallel}\right) < 0 \tag{3.72}$$

for small $k_\parallel/k_\perp$ (almost perpendicular propagation). Equation 3.71 is simply the condition of the hose instability indicating that the hose instability occurs also for the compressional mode. Equation 3.72, which is satisfied for $\beta_\perp > \beta_\parallel \sim 1$, is a complementary situation to the hose instability. The instability represented by equation 3.72 is called the mirror instability.

Let us briefly discuss the physical implication of the latter instability. As can be seen from the condition, equation 3.69, which was necessary to derive the unstable solution, the mirror instability is *not* the instability of the compressional Alfvén wave whose dispersion relation is given by $\omega^2 k^2 v_A{}^2 \cong 1$. In fact, as can be seen from equation 3.70, ω is purely imaginary, either growing or decaying. Such a mode is called the entropy mode. The mirror instability is hence the instability of a compressional entropy mode. This mode appears also as a drift wave when diamagnetic drift motion is considered (see the next section). The Im $\omega < 0$ solution of equation 3.70 for a plasma with an isotropic pressure, $\beta_\perp = \beta_\parallel$, represents transit-time damping [*Stix*, 1962] of the entropy mode. Transit-time damping is the magnetic analog of Landau damping where $\mu B_\parallel$ acts like an electrostatic potential φ, where μ is the magnetic moment. When a compressional wave is set up, it is ordinarily damped out by the transit-time damping. However, when $\beta_\perp > \beta_\parallel \sim 1$, the diamagnetic repulsion of the plasma, which is trapped in the local mirror field created by the wave, excludes the magnetic field and further decreases the parallel component of the local magnetic field. This accelerates the flow of plasma into the thus deepened well of the local mirror, and therefore the perturbation grows.

In the magnetosphere, except possibly in the tail, $\beta_\perp$ in general is larger than $\beta_\parallel$; hence the mirror instability is more likely to occur.

3.1e. Instabilities in partially ionized plasmas. When the plasma is only partially ionized, such as in the ionosphere, the difference in ν/ω_c (ν is the collision frequency between particular species and neutrals) for electrons and ions can produce a difference in drift velocities among those species when an external electric field in applied. Such a situation brings about some unique instabilities in the partially ionized plasma. In this subsection we treat such instabilities. Although velocity-space instabilities are treated exclusively in this subsection, we introduce for the first time the effect of coordinate-space nonuniformity. This is because the existence of an electric field in a partially ionized plasma, which is necessary to drive a drift, is often related to nonuniformity in the plasma.

We consider a model plasma, which is imbedded in uniform magnetic and electric fields $\mathbf{B}_0$ and $\mathbf{E}_0$. The plasma has its density gradient in the direction transverse to the ambient magnetic field. The coordinate system we take here is shown in Figure 7. The transverse electric field E_{0x}, in the direction of the density gradient, is either an ambipolar field or that plus a field applied from an external source. We consider a longitudinal electrostatic wave propagating in an arbitrary direction in such a plasma.

First, let us study the behavior of electrons. The density gradient and the drift may contribute to create an active conductivity (Re $\sigma < 0$). The necessary equations are the equation of motion

$$n\mathbf{v} = -\mu_e n(\mathbf{E} + \mathbf{v}\times\mathbf{B}) - D_e \nabla n \qquad (3.73)$$

and the equation of continuity

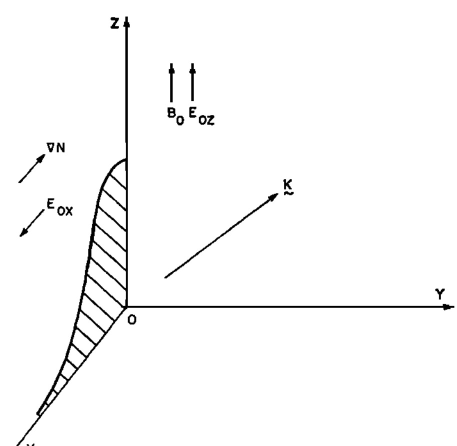

Fig. 7. Coordinate system used in the derivations of instability conditions for subsections 3.1e and 3.2a.

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{v}) = -\delta n \qquad (3.74)$$

where μ_e is the electron mobility $(=e/\nu_e m_e)$, D_e is the electron-diffusion constant $(=v_{Te}^2/\nu_e)$, ν_e is the electron-neutral collision rate, v_{Te} is the electron thermal velocity, and δ is the recombination frequency, which we assume to be negligibly small. The inertia term $\dot{\mathbf{v}}$ is ignored in equation 3.73 because we consider a low-frequency range where $\omega \ll \omega_{ce}$.

The unperturbed quantities, for which we use subscript O, are obtained from equations 3.73 and 3.74 as

$$v_{Oz} = -\mu_e E_{Oz} \qquad (3.75)$$

$$v_{Ox} = \frac{\kappa D_e - \mu_e E_{Ox}}{1 + \mu_e^2 B_0^2} \qquad (3.76)$$

$$v_{Oy} = \mu_e B_O v_{Ox} \qquad (3.77)$$

where κ shows the magnitude of the density gradient

$$\kappa = -\frac{d(\ln n_0)}{dx} > 0 \qquad (3.78)$$

For the perturbed quantities, we assume a phase factor of exp $i(\mathbf{k} \cdot \mathbf{x} - \omega t)$, and use subscript 1. The assumption of a longitudinal disturbance allows us to use a scalar potential φ_1 for the electric field

$$\mathbf{E}_1 = -i\mathbf{k}\varphi_1 \qquad (3.79)$$

Then we can solve equations 3.73 and 3.74 for the perturbed density n_1 in terms of the perturbed potential φ_1. Substituting the result into the definition of longitudinal conductivity shown in equation 2.30, we can obtain the electron conductivity as

$$\sigma_e = \frac{\omega\epsilon_0\omega_{pe}^2(ik_0^2 + \kappa k_y/\mu_e B_0)}{k^2[i\nu_e(\omega - \mathbf{k}\cdot\mathbf{v}_E) - v_{Te}^2 k_0^2]} \qquad (3.80)$$

$\mathbf{v}_E$ is the unperturbed velocity owing to the electric field, namely,

$$\mathbf{v}_E = -\mu_e E_{Oz}\mathbf{e}_z - \frac{\mu_e E_{Ox}}{1 + \mu_e^2 B_O^2}\mathbf{e}_x - \frac{\mu_e B_O \mu_e E_{Ox}}{1 + \mu_e^2 B_O^2}\mathbf{e}_y$$

and

$$k_O^2 = k_z^2 + \frac{k_x^2 + k_y^2}{1 + \mu_e^2 B_O^2}$$

The active range of σ_e can be immediately obtained from the condition that Re $\sigma_e < 0$, and is given by

$$0 < \omega < \mathbf{k}\cdot\mathbf{v}_E + \omega_e^* \qquad (3.81a)$$

where

$$\omega_e^* = \kappa v_T^2 k_y/\omega_{ce} \qquad (3.81b)$$

Thus, from equation 3.81*a*, we can see that both the electric field drift *and* the diamagnetic drift (due to the density gradient) contribute to make σ_e active, provided that the direction of the electric field is in its favor, i.e., if $\mathbf{k}\cdot\mathbf{v}_E > 0$ for the direction of k such that $\omega^* > 0$, namely if the y component of $\mathbf{v}_E$ is positive. For example, if we take the ambipolar electric field, which at the limit of $T_i = 0$ is given by

$$E_{0x} = \frac{\omega_e^* B_0}{k_y(1 + \mu_i\mu_e B_0^2)} > 0 \tag{3.82}$$

the y component of $\mathbf{v}_E$ becomes, when $\mu_e k_0 \gg 1$,

$$v_{Ey} = -\frac{E_{0x}}{B_0} < 0 \tag{3.83}$$

Therefore, the ambipolar electric field, which is directed in the positive x axis (i.e., $\mathbf{E}_0\cdot\nabla n_O < O$), has a stabliization effect (reducing the active nature of σ_e produced by ∇n_0). However, an externally applied electric field that is directed in the negative x axis ($\mathbf{E}_0\cdot\nabla\mathbf{n}_0 > 0$) enhances the instability [*Simon*, 1963]. On the other hand, any electric field in the z direction enhances the instability for a wave with $\mathbf{k}$ directed such that $\mathbf{k}\cdot\mathbf{E} > 0$.

Now, what kinds of instability will be expected to result from the active conductivity σ_e? We presume that cold ions in the plasma will constitute the passive conductivity; by assuming a quasi-neutral condition, we can ignore the conductivity of space (i.e., $-i\omega\epsilon_0$). First we can consider a low-frequency instability, where $\omega \ll \omega_{ci} \ll \nu_i$. For such a case, the ions constitute a simple resistive medium whose conductivity is given by

$$\sigma_i = \epsilon_O \omega_{pi}^2/\nu_i \tag{3.84}$$

where ω_{pi} is the ion plasma frequency. The condition for the instability is given from equations (2.33) and (2.35*a*) as $-\text{Re}\ (\sigma_e) > \epsilon_O\omega_{pi}^2/\nu_i$, at $\text{Im}\ (\sigma_e) = 0$, for $\omega \neq 0$; or, explicitly,

$$\frac{\omega_{pe}^2 k_O^2}{\nu_e k^2}\left(\frac{\omega_e^*\mathbf{k}\cdot\mathbf{v}_E}{k_O^4 D_e^2} - 1\right) - \frac{\omega_{pi}^2}{\nu_i} > 0 \tag{3.85}$$

When E_{Ox} is the ambipolar field, equation (3.85) reduces to the condition of the collisional helical instability as derived by *Kadomtsev* [1965]. Because the ambipolar electric field has a stabilizing effect ($k_y v_{Ey} < 0$), the instability condition, equation 3.85, is satisfied only in the presence of E_{Oz}. However, an instability is possible even in the absence of E_{Oz}, if E_{Ox} is applied externally. The related instabilities have been discussed by *Buneman* [1963] and by *Sato and Hatta* [1966] and are applied to the ionosphere by *Tsuda et al.* [1966].

Next, we consider a relatively higher frequency region: $\omega \gg \nu_i \gg \omega_{ci}$, but $\omega < \omega_{pi}$, where the ions constitute an inductive medium whose conductivity is given by

$$\sigma_i = \frac{i\epsilon_0\omega_{pi}^2}{\omega + i\nu_i} \tag{3.86}$$

If $\omega < \omega_{pi}$, we can still use the quasi-neutral assumption, where σ_i contributes to the passive conductivity σ_p. Because σ_p is now inductive (not resistive), either 'drift' or a 'density gradient' is sufficient to cause an instability. Let us first consider the effect of the density gradient and assume that there exists no electron drift parallel to B_O. The instability condition is obtained similarly

$$\frac{\mu_i\mu_e B_O^2}{1+\mu_i\mu_e B_O^2}\,\omega_e^* > k_y c_s \tag{3.87}$$

where c_s is the ion sound velocity $[= (m_e v_{Te}^2/m_i)^{1/2}]$. The related instabilities have been discussed by *Moiseev and Sagdeev* [1963].

We next consider the effect of drift alone. We neglect the density gradient and instead introduce a uniform electric field E_0 in the xz plane. The condition of instability is then

$$-\mu_e(E_{Oz}k_z + E_{Ox}k_y/\mu_e B_O) > (k_z^2 + k_y^2)^{1/2} c_s \tag{3.88}$$

Equation 3.88 represents the instability condition of an ion sound wave in a collisional plasma, in contrast to equation 3.18 for a collisionless plasma. It is interesting to note that in both cases the instability condition is given by v_0(= electron drift speed) $> c_s$.

3.2. Coordinate Space Instabilities

In this section we consider plasma instabilities for which a nonuniformity of the plasma distribution in space is necessary. We consider instabilities in which the driving free-energy source is entirely in the nonuniformity itself (often called universal instabilities because a plasma of a finite size is subject to this effect), as well as those in which the free energy is in some other agency, but requires the nonuniformity to convert it into driving energy for the instability.

One aspect of an instability of this kind has been shown in subsection 3.1*e*, where a density gradient is shown to produce a wave called the drift wave, whose frequency is given by $\omega^* = \kappa k_\perp v_T^2/\omega_c$.

3.2a. Drift-wave instabilities. In the presence of a density gradient, the equilibrium Vlasov equation for a particle with charge q and mass m takes the form

$$\mathbf{v}\cdot\frac{\partial f_0}{\partial \mathbf{x}} + \frac{q}{m}(\mathbf{v}\times\mathbf{B}_0)\cdot\frac{\partial f_0}{\partial \mathbf{v}} = 0 \tag{3.89}$$

As shown in Figure 7, we take the density gradient in the negative x direction (density decreases in the positive x direction). Then the general soluton of equation 3.88 can be written as

$$f_0(\mathbf{v}, \mathbf{x}) = f_{0v}(v_\perp, v_\parallel)\, g\left(x + \frac{v_y}{\omega_c}\right) \tag{3.90}$$

where the function g represents the space-dependent part of the distribution function. We assume the following form for g:

$$g\left(x + \frac{v_y}{\omega_c}\right) = 1 - \kappa\left(x + \frac{v_y}{\omega_c}\right) \tag{3.91}$$

where κ is the measure of the density gradient defined in equation 3.78.[1] If we assume an electrostatic perturbation, the perturbed distribution function f_1 obeys the following linearized Vlasov equation:

$$\frac{\partial f_1}{\partial t} + \mathbf{v}\cdot\frac{\partial f_1}{\partial \mathbf{x}} + \frac{q}{m}(\mathbf{v}\times\mathbf{B}_0)\cdot\frac{\partial f_1}{\partial \mathbf{v}} = \frac{q}{m}\left(\frac{\partial f_{0v}}{\partial \mathbf{v}}\cdot\frac{\partial \varphi_1}{\partial \mathbf{x}} - \frac{\kappa f_{0v}}{\omega_c}\frac{\partial \varphi_1}{\partial y}\right) \tag{3.92}$$

One can integrate equation 3.92 in the same manner as before. In particular when f_{0v} is given by a Maxwellian distribution

$$f_{0v}(v) = \frac{1}{(2\pi v_T^2)^{1/2}}\exp\left(-\frac{v^2}{2v_T^2}\right) \tag{3.93}$$

the corresponding electron and ion conductivities can be expressed using equations 3.7 and 2.30 [*Hasegawa*, 1968]:

$$\sigma_e = \frac{-i\omega\epsilon_0\omega_{pe}^2}{k^2 v_{Te}^2}\sum_{n=-\infty}^{\infty} I_n(\lambda_e)e^{-\lambda_e}\left[1 + \frac{\omega-\omega_e^*}{(2)^{1/2}k_\parallel v_{Te}} Z\left(\frac{\omega}{(2)^{1/2}k_\parallel v_{Te}}\right)\right] \tag{3.94a}$$

$$\sigma_i = \frac{-i\omega\epsilon_0\omega_{pi}^2}{k^2 v_{Ti}^2}\sum_{n=-\infty}^{\infty} I_n(\lambda_i)e^{-\lambda_i}\left[1 + \frac{\omega+\omega_i^*}{(2)^{1/2}k_\parallel v_{Ti}} Z\left(\frac{\omega}{(2)^{1/2}k_\parallel v_{Ti}}\right)\right] \tag{3.95a}$$

where Z is the plasma dispersion function defined in equation 3.29, I_n is the modified Bessel function of the nth order, and the argument of I_n is

$$\lambda = \left(\frac{k_\perp v_T}{\omega_c}\right)^2 \tag{3.96}$$

The drift-wave frequencies ω_e^* and ω_i^* have been defined in equation (3.81*a*).

When we consider a low frequency ($\omega \sim \omega_e^* \ll \omega_{ce}$) and long wavelength ($\lambda_e \ll 1$) range, the electron conductivity simplifies to

$$\sigma_e = \frac{-i\omega\epsilon_0\omega_{pe}^2}{k^2 v_{Te}^2}\left[1 + \frac{\omega-\omega_e^*}{(2)^{1/2}k_\parallel v_{Te}} Z\left(\frac{\omega}{(2)^{1/2}k_\parallel v_{Te}}\right)\right] \tag{3.94b}$$

Because Im $Z > 0$, Re σ_e becomes negative for $\omega < \omega_e^*$. That is, for ω less than the diamagnetic drift-wave frequency, the electron conductivity becomes active (negatively dissipative). The effect is quite similar to the two-stream case discussed in subsection 3.1*a* in that the Doppler-shifted wave undergoes inverse Landau damping, if we replace $k_0 v_0$ by ω_e^*.

Instability occurs by the coupling between the negative dissipation produced by the electrons and the ion drift wave. Because ion Landau damping contributes to stabilization, instability is possible for a wave with parallel phase velocity $\omega/k_\parallel$ much larger than the ion thermal speed, v_{Ti}. For this range of parallel phase velocity

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[1] Because $f_0 \geq 0$, κ must be sufficiently small that $g \geq 0$. More specifically, $|\kappa v_y/\omega_c| < 1$ for each species. The condition imposed on the value of x to make $g \geq 0$ can be eliminated by considering a suitable WKB treatment [*Mikhailovskii*, 1967], in which case the expansion of g shown in equation 3.91 is not needed. However, the results are identical to those of the present approach provided $k_\perp \gg \kappa$.

$$v_{Ti} \ll \frac{\omega}{k_\parallel} \ll v_{Te} \tag{3.97a}$$

$$\sigma_e = \frac{-i\omega\epsilon_0\omega_{pe}^2}{k^2 v_{Te}^2}\left[1 + i\,\frac{\omega - \omega_e^*}{k_\parallel v_{Te}}\left(\frac{\pi}{2}\right)^{1/2}\right] \tag{3.97b}$$

and

$$\sigma_i = \frac{-i\omega\epsilon_0\omega_{pi}^2}{k^2 v_{Ti}^2}\left[1 - \frac{\omega + \omega_i^*}{\omega}\, e^{-\lambda_i} I_0(\lambda_i)\right] \tag{3.95b}$$

The dispersion relation can be written, by assuming quasi-neutrality condition (i.e., neglecting the space dielectric constant)

$$\sigma_i + \sigma_e = 0 \tag{3.98}$$

From the imaginary part of equation 3.98 we can obtain the wave frequency ω as

$$\omega = \frac{\omega_i^* e^{-\lambda_i} I_0(\lambda_i)}{[1 + (T_i/T_e)] - e^{-\lambda_i} I_0(\lambda_i)} \tag{3.99}$$

and the instability condition is obtained from Re $\sigma < 0$ for ω given by (3.99):

$$\left(1 + \frac{\omega_i^*}{\omega_e^*}\right)e^{-\lambda_i} I_0(\lambda_i) - \left(1 + \frac{T_i}{T_e}\right) < 0 \tag{3.100}$$

Since electrons and ions have the same density gradient κ, $\omega_i^*/\omega_e^* = T_i/T_e$. Also $e^{-\lambda_i} I_0(\lambda_i) < 1$ for a finite value of the ion cyclotron radius $v_{Ti}/\omega_{ci}(\lambda_i \neq 0)$. Hence the instability condition is always satisfied. This means that whenever a plasma has a density gradient it becomes unstable for a wave whose parallel phase velocity is between the thermal velocities of ions and electrons. For this reason it is called the universal instability [*Moiseev and Sagdeev*, 1963].

When we apply this instability to the magnetosphere, we should be careful about the following two points. First is the effect of finite β. When $\beta \gtrsim m_e/m_i$, parallel motion of the particles bends the magnetic field and excites shear Alfvén wave. Then the above analysis must be modified to include the effect of the coupled shear mode. This was done by *Mikhailovskii* [1967]. The result shows that the condition of instability remains the same, but a condition arises that gives a maximum growth rate for $k_\parallel$, i.e., $k_\parallel v_A \sim (2)^{1/2}\omega_e^*$. When β is further increased, the drift wave instability can be shown to be stabilized completely when $\beta > 0.13$ by the ion Landau damping [*Kadomtsev*, 1965*b*].

The second point that is important in the magnetosphere is the effect of cold electrons. If cold electrons whose density is n_{0c} are mixed in, we have to add their contribution to equation 3.98. The conductivity of the cold electrons σ_c can be obtained from equation 3.27 by taking the limit as $T \rightarrow 0$

$$\sigma_c = i\omega\epsilon_0\left[\frac{\omega_{pc}^2}{\omega^2}\frac{k_\parallel^2}{k^2} + \frac{\omega_{pc}^2}{\omega^2 - \omega_{ce}^2}\frac{k_\perp^2}{k^2}\right] \tag{3.101a}$$

where ω_{pc} is the plasma frequency of the cold electrons. At $\omega \sim \omega_e^* \ll \omega_{ce}$ the second term can be shown to be negligible, and σ_c is given by

$$\sigma_c \sim i\omega\epsilon_0 \frac{\omega_{pc}^2}{\omega^2} \frac{k_\|^2}{k^2} \tag{3.101b}$$

Even for a relatively small density, ω_{pc} is very large. Hence so is σ_c and adding such a large σ_c to equation 3.98 completely changes the nature of the wave. This is because of the low inertia of the cold electrons, which can thus short-circuit the parallel electric field. If one works out the condition of the instability under this circumstance, by requiring a real ω solution from the real part of the dispersion relation, it is given by

$$\frac{n_{0c}}{n_0} < \frac{T_e}{T_i} \alpha \tag{3.102}$$

where n_0 is the hot plasma density and α is a number of the order of $(m_e/m_i)^{1/2}$ [*Hasegawa,* 1971*b*]. Equation 3.102 shows that the drift-wave instability produced by a hot plasma is stabilized by a fractional mixture of cold electrons.

Hence we conclude that the occurrence of the drift-wave instability in the magnetosphere is rather rare, either because of ion Landau damping for a high β situation, or the short-circuiting by cold electrons even for a low β situation.

Drift-wave instability of the compressional (magnetosonic) mode given by the dispersion relation equation 3.61*a* may, however, be possible. In the case of the compressional mode, the transit-time damping replaces the Landau damping of the electrostatic mode discussed above. Because the transit-time damping is proportional to β, the cold electrons do not contribute to the dispersion relation. Besides, the effect is manifested in high β situations where the electrostatic drift mode is stabilized. The drift-wave instability of this compressional mode has been treated by *Mikhailovskii and Fridman* [1967] and by *Hasegawa* [1971*c*].

3.2*b*. *Instability due to curved field lines: gravitational instability.* When the magnetic lines of force are not uniform, plasma trapped on such field lines may become unstable. When the field lines are straight, the instability is closely related to the one due to the nonuniformity in the plasma pressure shown in subsection 3.2*a*. Such a field configuration is inevitably a consequence of nonuniformity in pressure if the plasma and field are in pressure equilibrium, i.e.

$$\nabla[(B_0^2/2\mu) + nkT] = 0 \tag{3.103}$$

The instability associated with this case has been treated by *Krall and Rosenbluth* [1963] and more recently by *Mikhailovskii and Fridman* [1967].

When the field lines are curved, the particles moving parallel to the field lines see a centrifugal force. This force may produce an instability for the outer boundary of the plasma where the density decreases radially. This is equivalent to the gravitational instability of a heavy fluid on top of a light fluid, if we equate the centrifugal force with gravity. In the plasma, the instability is often called the flute instability because the most unstable mode has a flute-like perturbation, propagating perpendicular to the magnetic field (a field-aligned perturbation) [*Rosenbluth and Longmire,* 1957].

Let us consider a plasma with a similar geometry to that shown in Figure 7. We take the gravitational field $\mathbf{g}$ (which simulates the centrifugal force due to the particle motion parallel to the curved field lines) to be in the positive x direction, which corresponds to the direction of decreasing plasma density, i.e.

$$\mathbf{g}\cdot\nabla n_0 < 0 \tag{3.104}$$

We assume the plasma to be cold and collisionless and ignore dc electric fields. Now let us consider an electrostatic perturbation that propagates in the y, z plane. The linearized equation of motion for ions takes the form

$$-i\omega'\mathbf{v}_{\perp 1} = \frac{e}{m_i}(-i\mathbf{k}_\perp\varphi + \mathbf{v}_{\perp 1}\times\mathbf{B}_0) \tag{3.105}$$

where

$$\begin{aligned}\omega' &= \omega + kv_g \\ &= \omega + \frac{k_\perp g}{\omega_{ci}}\end{aligned} \tag{3.106}$$

and $k_\perp = k_y$, $k_\parallel = k_z$.

Solving equation 3.105 for $\omega \ll \omega_{ci}$

$$v_{\perp 1} = i\frac{\mathbf{B}_0\times\mathbf{k}_\perp}{B_0^2}\varphi - \frac{\omega'}{B_0\omega_{ci}}\mathbf{k}_\perp\varphi \tag{3.107}$$

Substituting this expression into the linearized equation of continuity, we have

$$\left(\frac{n_1}{n_0}\right)_{\text{ion}} = \varphi\left[\frac{\kappa k_\perp}{B_0\omega'} + \frac{e}{m_i}\left(\frac{k_\parallel^2}{\omega'^2} - \frac{k_\perp^2}{\omega_{ci}^2}\right)\right] \tag{3.108a}$$

where

$$\kappa = -\frac{d\ln n_0}{dx} > 0$$

For electrons, because the gravitational force produces little drift (smaller by the mass ratio than that of the ions, although the electron drift can become comparable to the drift of the ions if centrifugal force is used), we have

$$\left(\frac{n_1}{n_0}\right)_{\text{electrons}} = \varphi\left(\frac{\kappa k_\perp}{B_0\omega} - \frac{e}{m_e}\frac{k_\parallel^2}{\omega^2}\right) \tag{3.108b}$$

The dispersion relation is obtained by assuming quasi-neutrality (ignoring the vacuum dielectric constant) as

$$\frac{\kappa\omega_{ci}}{k_\perp}\left(\frac{1}{\omega + k_\perp g/\omega_{ci}} - \frac{1}{\omega}\right) = 1 - \frac{m_i}{m_e}\left(\frac{k_\parallel\omega_{ci}}{k_\perp\omega}\right)^2 \tag{3.109}$$

Now if we assume an ideal flute mode such that $k_\parallel = 0$, the unstable solution is easily obtained by assuming $\omega \gg k_\perp g/\omega_{ci}$ and expanding the first term of the left-hand side of expression 3.109 in powers of $k_\perp g/(\omega_{ci}\omega)$ as

$$\omega^2 + g\kappa = 0 \tag{3.110a}$$

or

$$\omega = \pm i(g\kappa)^{1/2} \tag{3.110b}$$

Equation 3.110*a*, *b* corresponds to the growth rate of the classical gravitational instability.

However, unlike classical fluids, the anisotropic nature of the plasma conductivity produces an interesting stabilization effect if $k_\parallel \neq 0$. For example, as in the case of magnetospheric plasma, if the field lines are connected to a conductive region perpendicular to themselves (the ionosphere), the charge separation perpendicular to the field lines that drives the instability is short-circuited by the large electron conductivity in the parallel direction in a way similar to the case of the drift-wave instability. This effect is represented by the second term on the right-hand side of expression 3.109. One can see from the previous argument that the unstable solution disappears when the right-hand side changes its sign. The stability condition then is obtained roughly as

$$\frac{k_\parallel}{k_\perp} > \left(\frac{m_e}{m_i}\right)^{1/2} \frac{(g\kappa)^{1/2}}{\omega_{ci}} \tag{3.111}$$

Equation 3.111 shows that only a perturbation having small perpendicular wavelength (large $k_\perp$) grows under such circumstances.

On the other hand, it is known that the gravitational instability is stabilized for a short perpendicular wavelength compared to the ion cyclotron radius [*Lehnert,* 1961] because of the neutralization of charge separation due to the finite size of the ion cyclotron radius. The stabilization condition due to this finite cyclotron-radius effect can be obtained by using the Vlasov equation as in subsection 3.2*a*, including the gravitational force. The stabilization condition then reads

$$g\kappa < (\omega_i^*)^2/4 \tag{3.112a}$$

where ω_i^* is the ion drift-wave frequency. Using $g = T_i/m_iR$ where R is the radius of the curvature of the field line, and T_i is the ion temperature in energy units, equation 3.112*a* can be expressed also as

$$k_\perp^2\rho_i^2 > 4/\kappa R \tag{3.112b}$$

where $\rho_i(= v_{Ti}/\omega_{ci})$ is the ion Larmor radius. Therefore, the gravitational instability is stabilized for a long wavelength perturbation by the short-circuiting of electrons moving rapidly parallel to the field lines, and for a short wavelength perturbation by the finite cyclotron radius effect of ions.

If one can assume a model in which the ionosphere is a perfect conductor at a low frequency regime ($\omega \ll \omega_{ci}$), one can see that by combining equations 3.111 and 3.112*b*, the gravitational instability is stabilized for a perturbation of *any* size if

$$Rk_\parallel > 2\left(\frac{m_e}{m_i}\right)^{1/2} \tag{3.113}$$

The parallel wave number $k_\parallel$ is lower-bounded by π over twice the length of the field line ($\sim \pi/LR_E$ where L is the equatorial crossing distance in units of earth radii, R_E). Thus $k_\parallel > \pi/LR_E$, whereas the radius of curvature $R \sim LR_E/3$. Hence equation 3.113 is always satisfied. This result indicates that the magnetospheric plasma may be stable against the gravitational instability. (In reality, however, because the cold electron density n_{0c} is different from that of the hot proton density n_0, the right-hand side has to be multiplied by $(n_0/n_{0c})^{1/2}$. Also note that we have assumed that the ionosphere is a perfect conductor perpendicular to B_0.)

3.2c. Kelvin-Helmholtz instability. While the drift-wave instability arises purely because of the nonuniformity of a plasma, the gravitational instability occurs because of the nonuniformity plus the gravitational field. The latter is the analogy of the classical Rayleigh-Taylor instability. Another classical instability similar to this is the Kelvin-Helmholtz instability. This instability requires nonuniformity plus a shear flow of fluid. Such an instability also occurs in a plasma and may be applicable to the magnetospheric boundaries or the charged layer in the aurora's sheet.

Here we consider an example of the electrostatic Kelvin-Helmholtz instability produced by a shear flow due to an E (nonuniform) cross B_0 (uniform) drift of a plasma. The nonuniform electric field is a consequence of a nonneutral, charged sheet as shown in Figure 8. As an example, we take an electron sheet with thickness $2a$ placed with its surface parallel to the uniform magnetic field. Because of the negative charge of electrons, a steady electric field is directed toward the center of the sheet and produces a shear flow $v_0(x)$ $(= E_0(x)/B_0)$ having a value at $x = \pm a$

$$v_0(a) \equiv v_0 \qquad v_0(-a) \equiv -v_0 \tag{3.114}$$

We will now consider the surface wave created by the surface charge if we have an undulating boundary at $x = \pm a$. The surface wave is generated only by the charges at the surface. The electrostatic-field equation is thus the Laplace equation for the electrostatic potential φ_1,

$$\nabla^2\phi_1 = 0 \tag{3.115}$$

meaning that the wave we consider is an incompressible mode. Equation 3.115 has a general solution for a two-dimensional system such as that shown in Figure 8, given by

$$\varphi_1 \sim e^{i(ky-\omega t)}\cdot e^{\pm kx} \tag{3.116a}$$

Hence we choose

$$\begin{aligned} \varphi_1(x > a) &= Ae^{i(ky-\omega t)}e^{-kx} \\ \varphi_1(-a < x < a) &= e^{i(ky-\omega t)}(Be^{kx} + Ce^{-kx}) \\ \varphi_1(x < -a) &= De^{i(kx-\omega t)}e^{kx} \end{aligned} \tag{3.116b}$$

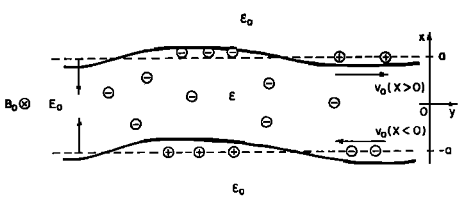


Fig. 8. Negatively charged sheet in a magnetic field B_0. The nonuniform electric field in the sheet generates a shear flow $\pm v_0$ at $x = \pm a$. A perturbation at the surface gives rise to a surface charge that leads to the Kelvin-Helmholtz instability.

The boundary conditions at $x = \pm a$ are (1) the continuity of the tangential electric field $E_y = \partial\varphi_1/\partial y$ and (2) the discontinuity of the normal electric field by the amount of the surface charge ρ_s, i.e.

$$\begin{aligned} \left.\frac{\partial\varphi_1}{\partial x}\right|_{x\to a^+} - \left.\frac{\partial\varphi_1}{\partial x}\right|_{x\to a^-} &= -\frac{\rho_s(a)}{\epsilon_0} \\ \left.\frac{\partial\varphi_1}{\partial x}\right|_{x\to -a^+} - \left.\frac{\partial\varphi_1}{\partial x}\right|_{x\to -a^-} &= -\frac{\rho_s(-a)}{\epsilon_0} \end{aligned} \tag{3.117}$$

where ρ_s can be obtained from the equation of continuity

$$\frac{\partial n_1}{\partial t} + \nabla\cdot(n_1\mathbf{v}_0 + n_0\mathbf{v}_1) = 0 \tag{3.118}$$

Following *Buneman et al.* [1966], we consider here a low-frequency perturbation such that $\omega \ll \omega_{ce}$. Then the perturbed velocity $\mathbf{v}_1$ in equation 3.118 can be expressed simply by

$$\mathbf{v}_1 = \frac{\mathbf{E}_1 \times \mathbf{B}_0}{B_0^{\,2}} \tag{3.119}$$

We consider an electrostatic perturbation, $\nabla \times \mathbf{E}_1 = 0$, hence from equation 3.119, $\nabla\cdot\mathbf{v}_1 = 0$. Equation 3.118 then can be reduced to

$$\frac{\partial n_1}{\partial t} + \mathbf{v}_0\cdot\nabla n_1 = -\mathbf{v}_1\cdot\nabla n_0 \tag{3.120}$$

By substituting equation 3.119 into equation 3.120 we have for the number-density perturbation n_1

$$n_1(\pm a) = \frac{k\varphi_1}{\omega \mp kv_0}\frac{1}{B_0}\frac{\partial n_0}{\partial x} \tag{3.121}$$

Now, if the charged sheet has a sharp boundary at $x = \pm a$ as assumed, $n_0(x)$ has the form of a unit step function $U(x)$, which may be written

$$n_0(x) = n_0[U(x + a) - U(x - a)] \tag{3.122}$$

then

$$\frac{\partial n_0}{\partial x} = n_0[\delta(x + a) - \delta(x - a)] \tag{3.123}$$

Hence, the surface charge density ρ_s can be obtained from equations 3.121 and 3.123:

$$\rho_s(\pm a) = \pm\frac{en_0}{B_0}\frac{k\varphi_1}{\omega \mp kv_0} \tag{3.124}$$

The dispersion relation can be obtained by applying boundary conditions that are given by equation 3.117, and the continuity of φ_1, to the solution of the Laplace equation (3.116*b*);

$$\frac{4\omega^2}{\omega_0^{\,2}} = \left(1 - \frac{2kv_0}{\omega_0}\right)^2 - e^{-4ka} \tag{3.125}$$

where

$$\omega_0 = \frac{en_0}{\epsilon_0 B_0} = \frac{\omega_{pe}^2}{\omega_{ce}}$$

The dispersion relation derived here has an identical form to the one for an incompressible fluid [*Chandrasekhar,* 1961]. The instability occurs when the right-hand side of equation 3.125 is negative, or for the wave number k satisfying

$$2ka \lesssim 1.3 \qquad (3.126)$$

Therefore, the instability occurs when the wavelength in the direction of the shear flow is comparable to or longer than 2π times the width of the charged sheet $2a$. The consequence of the instability is the deformation of the sheet into periodic curls [*Hallinan and Davis,* 1971] around the magnetic lines of force. Since the instability of a charged sheet grows for the wave propagating in the direction of the shear flow, for an electron sheet the curls produced by the instability will be in the clockwise direction looking in the direction of the magnetic field. On the other hand, for a sheet of positive ions, although the direction of the shear flow is given by the same formula, $\mathbf{E} \times \mathbf{B}$, the direction of E is reversed; hence so is the direction of the curls. For either case, the curls are formed in the direction of the cyclotron motion of the particles (see subsection 4.3).

The Kelvin-Helmholtz instability, which may be applicable to the magnetospheric boundaries, has a different character, although in principle the instability is the same kind. In the case of the magnetospheric boundary, one has to consider two plasmas with different flow velocity that are in contact at a surface parallel to the flow. The excited mode is the Alfvén wave. Here I do not introduce the details of this case but suggest references [*Boller and Stolov,* 1970; *Sen,* 1965; *Southwood,* 1968; *Talwar,* 1964].

3.2d. Instabilities of current pinch. In this subsection, we consider instabilities driven by a current in a plasma of finite cross section. As we have seen in subsection 3.1*a*, a current produces a two-stream instability even in a uniform plasma. In that case, the threshold velocity of the electrons was given by the ion sound velocity. For a current in a plasma with a finite cross section, extra free energy is available from the nonuniformity in space; therefore the threshold of the instability can become lower. One such example was shown in subsection 3.1*e* for the electrostatic mode in a collisional plasma.

First let us consider a plasma with circular cross section. Three kinds of deformation of such a circular pinch are considered in Figure 9. The first case, case *a*, is called sausage-type instability. The azimuthal magnetic field $B_{0\theta}$ produced by the current I_0 becomes stronger at the neck point because $B_{0\theta} \sim \mu_0 I_0/2\pi r$; hence the perturbation tends to grow. The threshold condition for the instability is obtained as follows. For a shear mode, plasma moves with the magnetic field; hence the total flux inside the plasma is constant

$$r^2 B_{0z} = \text{const} \qquad (3.127a)$$

where B_{0z} is the axial magnetic field. The change in B_{0z} associated with the change in radius r of the plasma column is then given by

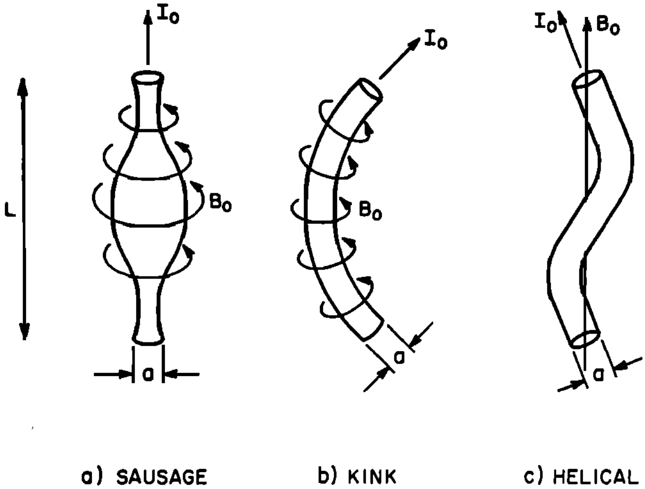


Fig. 9. Various shapes of instabilities of cylindrical current pinches.

$$\delta B_{0z} = -B_{0z} \cdot \frac{2dr}{r} \tag{3.127b}$$

On the other hand, the associated change in the azimuthal magnetic field $B_{0\theta}$ is given by

$$\delta B_{0\theta} = -B_{0\theta} \frac{dr}{r} \tag{3.128}$$

because the current I_0 is constant. The total change of the magnetic-field pressure directed inward is then

$$\begin{aligned} \delta p &= \delta\left(\frac{B_{0\theta}^2}{2\mu_0} - \frac{B_{0z}^2}{2\mu_0}\right) \\ &= -\left(\frac{B_{0\theta}^2}{\mu_0} - \frac{2B_{0z}^2}{\mu_0}\right)\frac{dr}{r} \end{aligned} \tag{3.129}$$

The instability condition is simply that the change in the magnetic pressure associated with an increase of radius dr is negative. Thus the pinch is unstable against the sausage-type perturbation when

$$B_{0\theta}^2 > 2B_{0z}^2 \tag{3.130}$$

The stability condition for the kink-type perturbation shown in Figure 9*b* can be obtained in the same way, and the condition of the instability for such a case becomes

$$B_{0\theta}^2 \ln\left(\frac{L}{a}\right) > B_{0z}^2 \tag{3.131}$$

where L and a are the length and the radius, respectively, of the current pinch.

In the magnetosphere, field-aligned currents have often been observed during substorm times [e.g., *Zmuda et al.*, 1966; *Cummings and Dessler*, 1967; *Cloutier et al.*, 1970]. However, the maximum horizontal field $B_{0\theta}$ produced by this current is of the order of $10^3\ \gamma (\gamma = 10^{-5}$ gauss), which is less than one-tenth of the geomagnetic field B_{0z}. Therefore, the instabilities of type *a* and *b* discussed above are unlikely to occur.

However, the instability of type *c* in Figure 9, the helical type (also called kink instability) is quite likely to occur because of the much lower threshold. The threshold condition of the helical-type instability will be shown later to be

$$B_{0\theta} > \frac{2\pi a}{L} B_{0z} \tag{3.132a}$$

While the instabilities of type *a* and *b* are primarily due to the pinch effect (compressional effect) of the current-generated magnetic field, the type *c* instability is due to the tension of the field lines that are bent into a helical shape by the current. When the tension of the bent field lines causes them to tend to straighten, the current tends to deform them into a helical shape.

Let us derive here the condition of the instability, equation 3.132*a*, following *Kodomtsev* [1966]. We assume that the plasma is collisionless and hence has an infinite conductivity. In such a case the current flows only at the surface of the column. Furthermore, we assume an incompressible perturbation (perturbation corresponding to the shear Alfvén mode). Because we consider only a long-wavelength and low-frequency perturbation, we can use MHD equations. If we assume incompressibility, the density perturbation $\rho_1 = 0$ and the velocity perturbation satisfy $\nabla \cdot \mathbf{v}_1 = 0$; the necessary set of MHD equations for perturbed quantities are given by

$$m_i n_0 \frac{d\mathbf{v}_1}{dt} = \mathbf{J}_1 \times \mathbf{B}_0 - \nabla p_1 \tag{3.133}$$

$$\nabla \times \mathbf{B}_1 = \mu_0 \mathbf{J}_1 \tag{3.134}$$

$$\mathbf{E}_1 + \mathbf{v}_1 \times \mathbf{B}_0 = 0 \tag{3.135a}$$

$$\nabla \times \mathbf{E}_1 = -\frac{\partial \mathbf{B}_1}{\partial t} \tag{3.136}$$

where we have used the same symbol identifications as before. In equation (3.133), the term $\mathbf{J}_0 \times \mathbf{B}_1$ is dropped because of the assumption that the dc current is concentrated only at the surface. The total derivative of $\mathbf{v}_1$ in the same equation has to be identified as equal to the partial derivative because the plasma as a whole is not moving. Then if we use a displacement vector $\boldsymbol{\xi}$ instead of the velocity $\mathbf{v}_1$ defined by

$$\mathbf{v}_1 = \partial \boldsymbol{\xi} / \partial t \tag{3.137}$$

equations 3.133 to 3.136 reduce to

$$\left(-\omega^2 m_i n_0 + \frac{k^2 B_0^2}{\mu_0}\right)\boldsymbol{\xi} = -\nabla\left[\frac{\mathbf{B}\cdot\mathbf{B}_0}{\mu_0} + p_1\right] \tag{3.138}$$

while from equations 3.135*a* and 3.136

$$\mathbf{B} = \nabla \times (\boldsymbol{\xi} \times \mathbf{B}_0) = (\mathbf{B}_0 \cdot \nabla)\boldsymbol{\xi} \tag{3.139}$$

if $\nabla \cdot \boldsymbol{\xi} = 0$ (incompressibility) and $B_0 = \text{const}$ ($J_0 = 0$ inside the plasma). In deriving equation 3.138, we assumed an e^{ikz} dependency on z (axial direction). The boundary conditions needed for the problem can be obtained as follows. If we write the external magnetic field and its perturbation by B_0^e and B_1^e respectively, the condition of total pressure balance gives

$$p_0 + p_1 + \frac{1}{2\mu_0}(B_0^2 + B_1^2) = \frac{1}{2\mu_0}[(B_0^e)^2 + (B_1^e)^2] \tag{3.140}$$

In our case the unperturbed pressure p_0 is constant in space and time, and hence the perturbed pressure is zero because of the incompressibility. We evaluate equation 3.140 at a displaced boundary at ($\mathbf{r} = \mathbf{r}_0 + \boldsymbol{\xi} = \mathbf{r}_0 + \xi_n\mathbf{n}$) to obtain the boundary condition for the perturbed equations, where $\mathbf{n}$ is the normal vector at the surface, and ξ_n is the normal displacement. Expanding equation 3.140 into powers of the perturbed quantities and retaining linear terms, we obtain one of the boundary conditions

$$\frac{\mathbf{B}_0 \cdot \mathbf{B}_1}{\mu_0} = \frac{\mathbf{B}_0^e \cdot \mathbf{B}_1^e}{\mu_0} + \frac{\xi_n}{2\mu_0}\left[\frac{\partial (B_0^e)^2}{\partial n} - \frac{\partial B_0^2}{\partial n}\right] \tag{3.141}$$

The second boundary condition is obtained from the vanishing tangential component of the electric field seen by the plasma because of the assumed perfect conductivity of the plasma, that is

$$\mathbf{E}_{1t} + (\mathbf{v}_1 \times \mathbf{B}_0^e)_t = 0 \tag{3.142a}$$

where subscript t indicates the tangential components. Using equations 3.136 and 3.137, the above expression can be reduced to

$$\mathbf{n} \cdot \mathbf{B}_1^e = \mathbf{n} \cdot \nabla \times (\boldsymbol{\xi} \times \mathbf{B}_0^e) \tag{3.142b}$$

Now let us solve equation 3.138, subject to the boundary conditions given by equations 3.141 and 3.142b, for a plasma column of a circular cross section with radius a. If we take the divergence of equation 3.138, because we assume an incompressible perturbation, $\nabla \cdot \boldsymbol{\xi} = 0$, and noting that $p_1 = 0$

$$\nabla^2\left(\frac{\mathbf{B} \cdot \mathbf{B}_0}{\mu_0}\right) = 0 \tag{3.143}$$

which can readily be solved, and

$$\frac{\mathbf{B} \cdot \mathbf{B}_0}{\mu_0} = A \cdot \frac{I_n(kr)}{I_n(ka)} e^{i(n\theta + kz)} \tag{3.144}$$

where I_n is the modified Bessell function of the first kind and A is the integration constant designating the value of $\mathbf{B} \cdot \mathbf{B}_0/\mu_0$ at $r = a$. The radial displacement ξ_r can be obtained by substituting equation (3.144) into (3.138)

$$\xi_r(r) = \frac{\mu_0 k}{\omega^2 m_i n_0 - k^2 B_0^2} A \frac{I_n'(kr)}{I_n(ka)} \tag{3.145}$$

Outside the plasma, we have $\nabla \times \mathbf{B}_1^e = 0$, as well as $\nabla \cdot \mathbf{B}_1^e = 0$; hence $\mathbf{B}_1^e$ can be expressed by $\nabla\psi$, where $\nabla^2\psi = 0$. The solution of ψ bound at infinity is

$$\psi = \frac{CK_n(kr)}{K_n(ka)} \tag{3.146}$$

where K_n is the modified Bessell function of the second kind, and C is the integration constant.

Now we use the boundary conditions. First we take the pressure balance condition, equation 3.141. Outside the plasma, the unperturbed magnetic field $\mathbf{B}_0^e$ has axial and azimuthal components B_{0z}^e and $B_{0\theta}^e$. Since B_{0z}^e is assumed uniform, while $B_{0\theta}^e \sim 1/r$

$$\frac{\partial}{\partial r}[(B_{0z}^e)^2 + (B_{0\theta}^e)^2] = -2[B_{0\theta}^e(a)]^2/a$$

at the boundary $r = a$. Then equation 3.141 gives

$$A = \frac{i}{\mu_0}\left(kB_{0z}^e + \frac{n}{a}B_{0\theta}\right)C - \frac{(B_{0\theta}^e)^2}{\mu_0 a}\xi_r(a) \tag{3.147}$$

In a similar way, the other boundary condition, that of vanishing tangential electric field given by equation 3.142b, gives

$$i\left(kB_{0z}^e + \frac{n}{a}B_{0\theta}^e\right)\xi_r(a) = Ck\frac{Kn'(ka)}{Kn(ka)} \tag{3.148}$$

Combining equations 3.145, 3.147, and 3.148, and eliminating $\xi_r(a)$, A, and C, we obtain the following dispersion relation

$$\mu_0 m_i n_0 \omega^2 = k^2B_0^2 - \left(kB_{0z}^e + \frac{n}{a}B_{0\theta}^e\right)^2\frac{I_n'(ka)K_n(ka)}{I_n(ka)K_n'(ka)} - \frac{(B_{0\theta}^e)^2k}{a}\frac{I_n'(ka)}{I_n(ka)} \tag{3.149a}$$

For instability, the right-hand side has to give a negative value. The first and the second term are positive because $K_n/K_n' < 0$, while $I_n'/I_n > 0$. Hence it is the last term on the right-hand side that gives rise to an unstable solution. This negative contribution originates from the fact that $(\partial/\partial r)$ $(B_{0e}^2) < 0$; namely that the external magnetic field pressure decreases against radial displacement of the plasma column.

For field-aligned currents in the magnetosphere, we can assume $B_{0z}^e \gg B_{0\theta}^e$. In this case a long wavelength perturbation characterized by $ka \ll 1$ can lead to an unstable solution. At small ka, $I_n'/I_n = n/ka$, $K_n'/K_n = -n/ka$; thus the dispersion relation reduces to

$$\mu_0 m_i n_0 \omega^2 = k^2B_0^2 + \left(kB_{0z}^e + \frac{n}{a}B_{0\theta}^e\right)^2 - \frac{n(B_{0\theta}^e)^2}{a^2} \tag{3.149b}$$

One can see from equation 3.149b that a perturbation of the azimuthal mode with $n = 1$ is unstable (helical perturbation), and the condition of instability can be found as $|k| > B_{0\theta}/a\, B_{0z}$ or

$$\frac{B_{0\theta}}{B_{0z}} > \frac{2\pi a}{L} \tag{3.132b}$$

the form presented before.

We now shift our interest to the pinch of an *infinitely extended sheet* current. In this case none of the above instabilities for cylindrical pinches is known to occur because all of those instabilities originate from radially decreasing magnetic field pressure, whereas in the case of a sheet current the external magnetic field generated by the current remains constant. We will now consider this problem in relation to the instability of the neutral sheet in the magnetospheric tail.

That a neutral sheet is subject to instability was first pointed out by *Dungey* [1958] and elaborated later by *Furth et al.* [1963]. The instability is called the tearing mode instability. Consider a sheet current that is infinitely extended in the yz plane and flowing in the y direction as shown in Figure 10. Such a current is sandwiched by the self-generated magnetic field in the z direction that pinches the current to an equilibrium size. One can assume a perturbation in the current $\mathbf{J}_1$ and the magnetic field $\mathbf{B}_1$ but can show that they produce a stably propagating wave if the plasma is assumed to be a perfect conductor. Only when the plasma has a finite resistivity does the instability set up. In this sense, the mechanism of the instability differs considerably from the instabilities of a cylindrical pinch. To show how the finite resistivity produces the instability, we rewrite one of the MHD equations, equation 3.135a, including the effect of finite resistivity, i.e.

$$\mathbf{B}_1 + \mathbf{v}_1 \times \mathbf{B}_0 - \eta \mathbf{J}_1 = 0 \tag{3.135b}$$

where η is the plasma resistivity in ohm meters. From this expression we can see that the effect of finite resistivity becomes important at the neutral layer at $x \sim 0$, where the z-directed magnetic field $B_0 \sim 0$. On the other hand, at distances sufficiently far from the neutral layer, the $\mathbf{v} \times \mathbf{B}$ term can dominate, and the plasma can be regarded as lossless. To understand the physical process of the instability, we hence choose a simple model in which the current layer is divided into two regions, that of resistivity at $|x| < \epsilon$, and that of no resistivity at $a > |x| > \epsilon$.

Let us first consider the dynamics in the resistive region, $|x| < \epsilon$. In this region $\mathbf{E}_1$ may be expressed as $\eta \mathbf{J}_1$ from equation 3.135a. Then using Maxwell's equations 3.134 and 3.136, we have

$$\frac{\partial \mathbf{B}_1}{\partial t} = \frac{\eta}{\mu_0} \nabla^2 \mathbf{B}_1 \tag{3.150a}$$

Equation 3.150a represents simply the skin effect of the plasma. For an eigenfunction of a form $e^{ik_z z}$, equation 3.150a gives a solution with a negative imaginary part of ω, indicating dissipation of wave energy and no instability. However, if the field solution at $|x| > \epsilon$ allows a solution, through boundary conditions, such

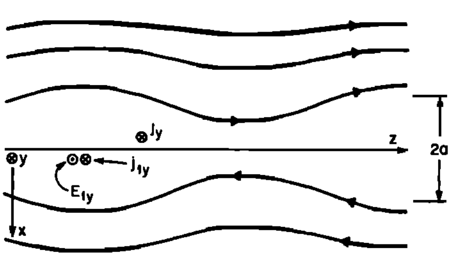


Fig. 10. Coordinate system used in derivation of instability condition for sheet current.

that $\mathbf{B}_1 \sim e^{\pm\kappa x}$ at $|x| < \epsilon$, a positive imaginary ω solution results. Anticipating such a case, we put, say for B_{1x} components, $B_{1x} \sim B_{1x}(x)e^{ikz+\gamma t}$, and equation 3.150$a$ becomes

$$\frac{d^2B_{1x}}{dx^2} - \left(k^2 + \frac{\gamma\mu_0}{\eta}\right)B_{1x} = 0 \tag{3.150b}$$

which can immediately be solved and

$$B_{1x} \sim A \cosh \left(k^2 + \frac{\gamma\mu_0}{\eta}\right)^{1/2} x \tag{3.151}$$

We now consider the lossless region. From equations 3.135a and 3.136 we can express the x component of velocity perturbation by B_{1x} as

$$ikv_{1x}B_0 = \gamma B_{1x} \tag{3.152}$$

where $B_0 = B_0(x)$ is the dc magnetic field produced by the sheet current.

Another equation that relates v_{1x} and B_{1x} can be obtained from equations 3.133 and 3.134; the pressure-gradient term is eliminated by taking the curl of equation 3.133. Assuming incompressibility, $\nabla\cdot\mathbf{v}_1 = 0$, and using $\nabla\cdot\mathbf{B} = 0$, we can derive the following equation

$$\frac{d^2B_{1x}}{dx^2} - \left(k^2 + \frac{B_0''}{B_0}\right)B_{1x} = \frac{-\gamma^2}{k^2v_A^2}\left(\frac{d^2}{dx^2} - k^2\right)\left(\frac{B_{1x}}{B_0}\right) \tag{3.153a}$$

which can further be reduced, for a small growth rate $\gamma^2 \ll k^2v_A^2$, to

$$\frac{d^2B_{1x}}{dx^2} - \left(k^2 + \frac{B_0''}{B_0}\right)B_{1x} = 0 \tag{3.153b}$$

where B_0'' is the second derivative of B_0 with respect to x. If B_0 is uniform so that $B_0'' = 0$, equation (3.153b) simply shows the electromagnetic cutoff mode in space. However, for a nonuniform sheet current confined within $|x| \leq a$, B_0''/B_0 can be seen to become negative. Equation 3.153b then admits a sinusoidal solution for small k. For example, if we write $B_0''/B_0 \sim -\lambda^{-2}$ the equation becomes

$$\frac{d^2B_{1x}}{dx^2} + \left(\frac{1}{\lambda^2} - k^2\right)B_{1x} = 0 \tag{3.153c}$$

or

$$B_{1x} = C \sin \left(\frac{1}{\lambda^2} - k^2\right)^{1/2} x \tag{3.154}$$

If we now connect this solution at $x = \epsilon$ with B_{1x} obtained for $|x| < \epsilon$ in equation 3.151, we can derive the growth rate γ as

$$\gamma = \eta/\epsilon^2\mu_0 \tag{3.155}$$

Although we cannot obtain an exact value of the growth rate from the above expression because ϵ is a quantity assumed in the derivation, we can understand the mechanism of the instability from the above argument. The driving force of the instability is the nonuniform magnetic field such that $B_0''/B_0 < 0$. The instability occurs for a wave length in the z direction longer than the thickness, i.e., $k < 1/\lambda \sim 1/a$, and for a plasma with a finite resistivity η. As a consequence of the instability, x-type neutral points tear the sheet current into a number of smaller segments.

The magnetospheric tail can hardly be considered as 'resistive'; therefore one might conclude that the tearing mode instability is not applicable there. However, if one takes into account the Landau damping caused by interactions between waves and warm particles, the tearing mode may become possible [*Coppi et al.*, 1966; *Hoh*, 1966].

3.2e. Resonant wave-particle interactions. In this final subsection of chapter 3, the nature of various wave-particle interactions and their relation to resonant-type plasma instabilities are presented. Some of them are strictly velocity-space effects, whereas others are coordinate-space effects. Hence they are slightly off the general topics of this section.

The importance of the various resonant frequencies for a particle trapped in the geomagnetic field has been discussed by *Dungey* [1964]. The linearized Vlasov equation that represents the particles trapped in the geomagnetic field has a form

$$\frac{\partial f_1}{\partial t} + \mathbf{v}\cdot\frac{\partial f_1}{\partial \mathbf{x}} + \frac{q}{m}(\mathbf{v}\times\mathbf{B}_0)\cdot\frac{\partial f_1}{\partial \mathbf{v}} = -\frac{q}{m}[\mathbf{E}_1 + \mathbf{v}\times\mathbf{B}_1]\cdot\frac{\partial f_0}{\partial v} \tag{3.156}$$

As was mentioned in subsection 3.1*b*, the Vlasov equation can be integrated along the trajectory of the particle and

$$f_1 = -\frac{q}{m}\int_{-\infty}^{t} [\mathbf{E}_1(t') + \mathbf{v}(t')\times\mathbf{B}_1(t')]\cdot\frac{\partial f_0}{\partial \mathbf{v}'}\,dt' \tag{3.157}$$

The unperturbed trajectory of a particle may be given by

$$x' \sim v_\parallel' t' + \vartheta_d t + \frac{v_\perp'}{\omega_c}\cos(\omega_c t' + \theta) \tag{3.158}$$

for

$$\begin{gathered} 2\pi(n+1) > \omega_b t' > 2\pi n \\ 2\pi(m+1) > \omega_d t' > 2\pi m \end{gathered} \tag{3.159}$$

$$x' \sim \frac{v_\parallel'}{\omega_b}\cos(\omega_b t' + \theta') + \frac{\vartheta_d}{\omega_d}\cos(\omega_d t' + \theta'')$$

for t from $-\infty$ to $2\pi n/\omega_b$ or from $-\infty$ to $2\pi m/\omega_d$ where m, n are integers. In equations 3.154 and 3.155, $v_\parallel' t'$ represents the motion parallel to $\mathbf{B}_0$ from the last bounce at the mirror point, $\vartheta_d t$ the gradient and curvature drifts (not the diamagnetic drift) from the last periodic longitudinal revolution, $v_\perp'/\omega_c \cos(\omega_c t' + \theta)$ the periodic cyclotron motion, $v_\parallel'/\omega_b \cos(\omega_b t' + \theta')$ the periodic bounce motion, and $\vartheta_d/\omega_d \cos(\omega_d t' + \theta'')$ the periodic longitudinal (azimuthal) motion around the earth.

If one assumes the space-time dependence $e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}$ and uses

$$e^{iz\cos\theta} \sim \sum_n J_n(z)e^{in(\theta-\pi/2)}$$

the integration over t along the trajectory produces qualitatively the following expression for f_1:

$$f_1 \sim \int \frac{F_0(v)\, d\mathbf{v}}{\omega - k_\parallel v_\parallel - k_\perp \hat{v}_d - l\omega_c} + \int \frac{G_0(v)\, d\mathbf{v}}{\omega - m\omega_b - n\omega_d} \tag{3.160}$$

where $F_0(v)$ and $G_0(v)$ are functions of velocity. Note here that the cyclotron frequency ω_c is not velocity-dependent, but that $\hat{v}_d$, the bounce frequency ω_b, and the longitudinal drift frequency ω_d are all velocity-dependent. When integration over $\mathbf{v}$ is performed along the real v axis but below the poles as was shown in subsections 3.1*a* and 3.1*c*, this resonant denominator produces the imaginary part of the dielectric constant. This represents the wave-resonant particle interaction. Depending on the nature of $F_0(\mathbf{v})$ or $G_0(\mathbf{v})$ at those resonant velocities, the direction of the energy flow (from the particle to the wave or from the wave to the particle) is decided. For example, we know already that from subsection 3.1*a* at $\omega \sim k_\parallel v_\parallel$ or from subsection 3.1*c* at $\omega - l\omega_c \sim k_\parallel v_\parallel$ such a resonance occurs. From the above expression, there are additional resonances at $\omega \sim k_\perp \hat{v}_d$, $\omega \sim m\omega_b$, and $\omega \sim n\omega_d$ that can in the same way contribute to produce resonant wave particle interactions. Note, however, that these resonances occur when a wave with a frequency ω is *given* a priori. Then particles whose velocity or energy agrees with the resonant conditions can exchange energy. This resonant condition *does not* determine the wave frequency itself (does not contribute to the real part of ϵ). Only when the particles are distributed very narrowly in energy does the resonant frequency determine the wave frequency. *Berk and Book* [1969] have shown that if ω_b is energy-independent (for example, a particle trapped in a parabolic electrostatic potential well), then the pole produced by $\omega = kv_\parallel$ (Landau damping) is canceled by part of the contribution from the second integral in equation 3.160, and only the $(\omega - m\omega_b)$ resonance remains to contribute to the real part of the dielectric constant. The energy-independent bounce motion, because of its precise periodicity, regenerates a wave that is at once Landau-damped. Also, as in the case of cyclotron waves, the bounce motion produces waves at multiples of the bounce frequency. However, in general, this does not apply for bounce motion in the magnetic field because ω_b is energy-dependent. It is shown also by *Berk and Pearlstein* [1971] that when the spread in ω_b due to the energy spread of the particles is larger than the average ω_b itself (i.e., if $\langle\omega_b^2\rangle > \langle\omega_b\rangle^2$), the regeneration does not occur and $m\langle\omega_b\rangle$ no longer contributes to the wave frequency through the real part of the dielectric constant. (A. Hasegawa and K. Nishihara have prepared data on instabilities associated with the bouncing particles in a magnetic mirror for publication in 1971.)

Finally note that the diamagnetic drift-wave frequency ω^* does not appear in the resonant denominator. This is because the diamagnetic drift is not the real drift of a particle. On the other hand, because the diamagnetic current does collectively contribute to the wave, ω^* determines the wave frequency (contributes to the real part of ϵ) and produces a new mode (drift wave).

3.3. *Summary*

A number of instabilities have been introduced in this part. To avoid possible confusion, we summarize here briefly the nature of those instabilities presented.

[3.1*a*] Electrostatic instabilities due to two-humped velocity distributions:

$$\mathbf{k} \parallel \mathbf{B}_0 \qquad \nabla \times \mathbf{E}_1 = 0$$

Necessary conditions for instability:

$$\frac{\partial f_0}{\partial v} > 0 \qquad \text{for} \qquad \omega \sim \omega_{pe}$$

$$v_p > c_s \qquad \text{for} \qquad \omega \lesssim \omega_{pi}$$

[3.1*b*] Electrostatic instabilities due to anisotropic velocity distributions:

$$k; \quad \text{arbitrary direction, } \nabla \times \mathbf{E}_1 = 0$$

Necessary conditions for instability:

$$\frac{T_\perp}{T_\parallel} > 1$$

or for $\omega \sim n\,\omega_c$

$$\frac{\partial f_{0\perp}}{\partial v_\perp} > 0$$

[3.1*c*] Electromagnetic instabilities due to anisotropic velocity distributions:

$$\mathbf{k} \parallel \mathbf{B}_0 \qquad \nabla \cdot \mathbf{E}_1 = 0$$

Necessary condition for instability:
Presence of current

$$\omega \sim \omega_c \quad \text{or} \quad \omega \sim 0$$

or

$$\frac{T_\perp}{T_\parallel} > \frac{\omega_c}{\omega_c - \omega}$$

[3.1*d*] Hydromagnetic instabilities due to anisotropic pressures:

$$k; \quad \text{arbitrary direction,}$$

$$\omega \ll \omega_{ci} \qquad k v_{T_i} \ll \omega_{ci} \qquad \nabla \cdot \mathbf{E}_1 \neq 0 \qquad \nabla \times \mathbf{E}_1 \neq 0$$

Hose instability:

$$1 - \sum_{\text{species}} \tfrac{1}{2}(\beta_\parallel - \beta_\perp) < 0$$

Mirror instability:

$$1 + \sum_{\text{species}} \beta_\perp \left(1 - \frac{\beta_\perp}{\beta_\parallel}\right) < 0$$

[3.1*e*] Instabilities in partially ionized plasmas:

$$k; \quad \text{arbitrary direction,} \quad \omega \lesseqgtr \omega_{ci} \qquad \nabla \times \mathbf{E}_1 = 0$$

Necessary condition for instability:

1. If $\omega_{ci}\tau_i \ll 1$ $\quad \mathbf{E}_0 \cdot \nabla n_0 > 0$
2. If $\omega_{ci}\tau_i \gg 1$ $\quad \omega_e^* > \mathbf{k} \cdot \mathbf{c}_s$ or $\mathbf{k} \cdot \mathbf{v}_E > \mathbf{k} \cdot \mathbf{c}_s$

[3.2*a*] Drift-wave instabilities:

∇n_0-type electrostatic drift wave mode, $\omega \sim \omega_e^*$

Necessary conditions of instability:

$$v_{Te} > \frac{\omega}{k_\parallel} > v_{Ti}$$

Stabilization due to cold electrons or high β effect.

[3.2*b*] Gravitational (flute, interchange, or Rayleigh–Taylor) instability:

$$\mathbf{k} \perp \mathbf{B}_0 \qquad \nabla \times \mathbf{E}_1 = 0$$

Necessary condition of instability:

$$\mathbf{g} \cdot \nabla n_0 < 0 \qquad \text{Im}\,\omega \propto (g\kappa)^{1/2}$$

Stabilization due to cold electrons if field lines are tied to the ionosphere.

[3.2*c*] Kelvin–Helmholtz instability: Only electrostatic ($\nabla \times \mathbf{E}_1 = 0$) case with $\mathbf{k} \perp \mathbf{B}_0$ is introduced, but electromagnetic case can also exist.
Necessary condition of instability:

$$2ka \leq 1.3$$

[3.2*d*] Instability of current pinch:

$$\omega \sim 0 \qquad \nabla \cdot \mathbf{E}_1 = 0$$

1. Cylindrical pinch
   a. Sausage type:

$$B_{0\theta}^2 > 2B_{0z}^2$$

   b. Kink type:

$$B_{0\theta}^2 \ln\left(\frac{L}{a}\right) > B_{0z}^2$$

   c. Helical:

$$B_{0\theta} > \frac{2\pi a}{L} B_{0z}$$

2. Sheet pinch

Tearing-mode instability:

$$\eta \neq 0 \qquad \text{Nonuniform current}$$

B. Review of Magnetospheric Plasma Instabilities

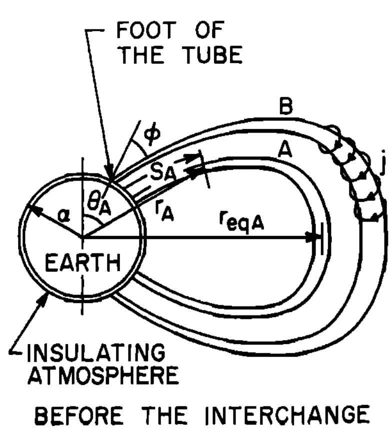
4. STABILITY OF PLASMAS IN VARIOUS PORTIONS IN THE MAGNETOSPHERE

In this chapter, we review papers published on the stability of plasmas in various portions of the magnetosphere and its periphery.

4.1. Stability of the Ring Current and Radiation Belt

4.1a. Coordinate space instabilities. The stability of the ring current (low-energy particles) and the radiation belt (high-energy particles) has been a classic subject in the magnetosphere since their discoveries. The questions raised here are why is the magnetospheric plasma so stable, and under what conditions may it become unstable? These are opposite the questions we ask of most laboratory devices. We divide the question into two groups by the nature of the possible instabilities, those in coordinate space or those in their velocity space. Because of its relatively large particle density, the ring-current particles, with an average energy of around 10 kev, act collectively. On the other hand, the radiation-belt particles, with energies of Mev, act more or less as a group of noninteracting single particles. Therefore, while the former group requires a self-consistent treatment, the latter does not, and a calculation of their wave-particle resonance with the wave carried by other groups of particles may be sufficient. This also means that the radiation belt particles may be mostly subject to velocity-space instabilities.

That the magnetospheric plasma is subject to the gravitational instability subsection 3.2*b* was first pointed out by *Gold* [1962] and *Axford and Hines* [1961]. It is called the interchange instability because the gravitational instability for plasmas in curved field lines causes interchange of the flux tubes (cf. Figure 11). The theory has been developed by *Sonnerup and Laird* [1963], who used MHD-type fluid assumptions. They have considered isothermal and adiabatic interchange of flux tubes, and concluded that the plasma is stable for isothermal interchange (no change in temperature during the interchange) but may be unstable for adiabatic interchange (no change in heat during the interchange) if the energy in the tubes of force decreases radially more rapidly than $r^{-4(\gamma-1)}$, where γ is the ratio of specific heats at constant pressure and constant



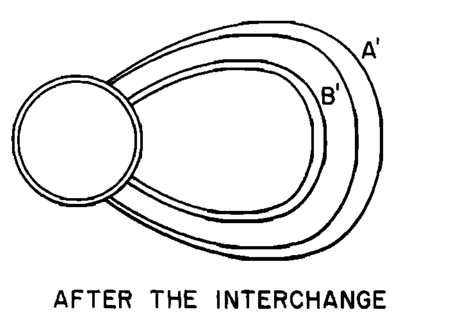


Fig. 11. Interchange of tubes of force of equal magnetic flux in the earth's magnetosphere (after *Sonnerup and Laird* [1963]).

volume. γ depends on the pitch-angle distribution, but on the average it takes a value $\sim 5/3$, leading to the critical radial decrease of energy of $r^{-8/3}$.

However, as was pointed out in subsection 3.2*b*, if the line of force is connected to a conductor, there exists a strong stabilizing effect. The ionosphere, though it is not a perfect conductor, can serve to short-circuit the charge separation needed to allow such interchange. *Chang et al.* [1965, 1966] have considered the effect of the ionosphere on the interchange instability using a phase-space treatment to overcome the difficulty of the MHD approximation in relation to the parallel electric field, and concluded that if adiabatic interchange is assumed, the critical radial dependence of the energy becomes r^{-7}. However, if the condition of adiabaticity is violated by a wave-particle resonance (such as mentioned in subsection 3.2*e*), the sufficient condition for *stability* is shown to be, considering the ionospheric conductivity,

$$\left(\frac{\Delta r}{R_E}\right)\frac{m}{L^4}\frac{n_i}{n_0} \gtrsim 0.3 \tag{4.1}$$

where $\Delta r \sim 50$ km, R_E (= earth's radius) $\sim 6{,}300$ km, m is the azimuthal mode number (integer), L is the radial distance in units of earth radii, n_i is the ionospheric plasma density, and n_0 is the proton density of the ring current. They concluded that the stability condition can be easily satisfied for the day side because of larger n_i, but may be marginal for the night side. Furthermore, as was pointed out in subsection 3.2*b*, stability is achieved if a fractional amount of cold electrons are intermixed. We thus conclude that the ring-current particles are very likely to be stable against the gravitational (interchange) instability.

Swift [1967*a*], taking the result of Chang et al., has pointed out that the interchange instability of the ring current may be responsible for auroral breakup. Although the energy gradient tends to become larger during the compressed state of the ring current, because of the arguments presented above, it is still rather unlikely that the instability can take place. In addition, the position of the ring current during storm time is compressed too far inward for it to be projected down to the auroral latitude.

Application of the drift-wave instability (subsection 3.2*a*) to the ring-current plasma has been considered by *D'Angelo* [1969]. However, as was mentioned in subsection 3.2*a*, because of the ion Landau damping in the high β plasma, as well as the cold electron short-circuiting, the instability is rather unlikely, except possibly at the plasmapause [*Hasegawa,* 1971*b*]. *Chamberlain* [1963] and *Liu* [1970] have considered the drift wave instability including the effect of gradient and curvature drifts. Usually, however, these instabilities have very small growth rates and may not be of much importance. Liu considered an electrostatic mode, whereas Chamberlain considered a noncompressional electromagnetic mode. However, as was shown by *Mikhaĭlovskiĭ and Fridman* [1967], the compressional electromagnetic mode may produce an important contribution to the instability of a high β plasma with a larger gradient in temperature than in density.

We again conclude that the ring-current plasma is stable against most of the driftmode instabilities, except possibly that of the compressional (magnetosonic) mode because this is the only mode that is excited in a high β plasma and that is not affected by cold electrons.

4.1b. Velocity space instabilities. In section 4.1, it was concluded that the ring-current plasma is very likely stable against coordinate-space instabilities. Hence the loss of those particles has to be produced by some other mechanism. We consider here the possibility of velocity-space instability. Whereas the coordinate-space instability, if it occurs, tends to directly smooth out the plasma nonuniformity, the velocity-space instability causes diffusion only in velocity space. However, it produces a pitch-angle scattering, and thus can also serve indirectly as a loss mechanism for the plasma.

That the electrons in the radiation belt may be subject to the cyclotron-wave instability because of their anisotropic pitch-angle distribution (subsection 3.1*c*) was first pointed out by *Brice* [1963]. The idea has been developed by *Kennel and Petschek* [1966] to calculate the limit of stably trapped particle fluxes with respect to the cyclotron instabilities. Kennel and Petschek have solved the velocity-space diffusion equation to obtain the anisotropy of the pitch-angle distribution, by assuming the noise amplitude of a whistler wave. The anisotropy of the pitch-angle distribution gives the growth rate of the whistler wave carried by cold electrons (subsection 3.1*c*). By assuming a suitable wave reflection at the ionosphere, they calculated the amplitude of the whistler wave using the growth rate obtained. Thus by matching the calculated amplitude with the originally assumed whistler amplitude, they could solve the chain of quasi-linear equations and obtain the stably trapped limit of electron flux (cf. Figure 12).

The relations needed to calculate the energy of the resonant particles have been obtained in their paper. The resonant condition for the cyclotron wave is given by

$$\omega - \omega_c = kv_\parallel \tag{4.2}$$

which applies for both electrons and protons by choosing the respective cyclotron frequencies ω_c. The energy of the resonant particles $E_R = mv_\parallel^2/2$ may then be expressed, using the dispersion relation of cold-electron- and proton-cyclotron waves (equations 3.48 and 3.49) with $f = \delta(v)$, as

$$\begin{aligned} E_{Re} &= E_B \frac{\omega_{ce}}{\omega}\left(1 - \frac{\omega}{\omega_{ce}}\right)^3 \\ E_{Ri} &= E_B\left(\frac{\omega_{ci}}{\omega}\right)^2\left(1 - \frac{\omega}{\omega_{ci}}\right)^3 \end{aligned} \tag{4.3}$$

where $E_B[= B_0^2/(2\mu_0 n_{0c})]$ is the magnetic field energy per particle of *cold* plasma that carries the wave. Furthermore, the ratio ω/ω_c can be expressed in terms of the anisotropy of the temperature of the resonating *hot* component (radiation-belt particles) from equation 3.59*a*

$$1 - \frac{\omega}{\omega_c} = \frac{T_\parallel}{T_\perp} \tag{3.59b}$$

If we substitute equation 3.59*b* into equation 4.3, we can express the energy of the resonating (hot) plasma in terms of the anisotropy of temperature as

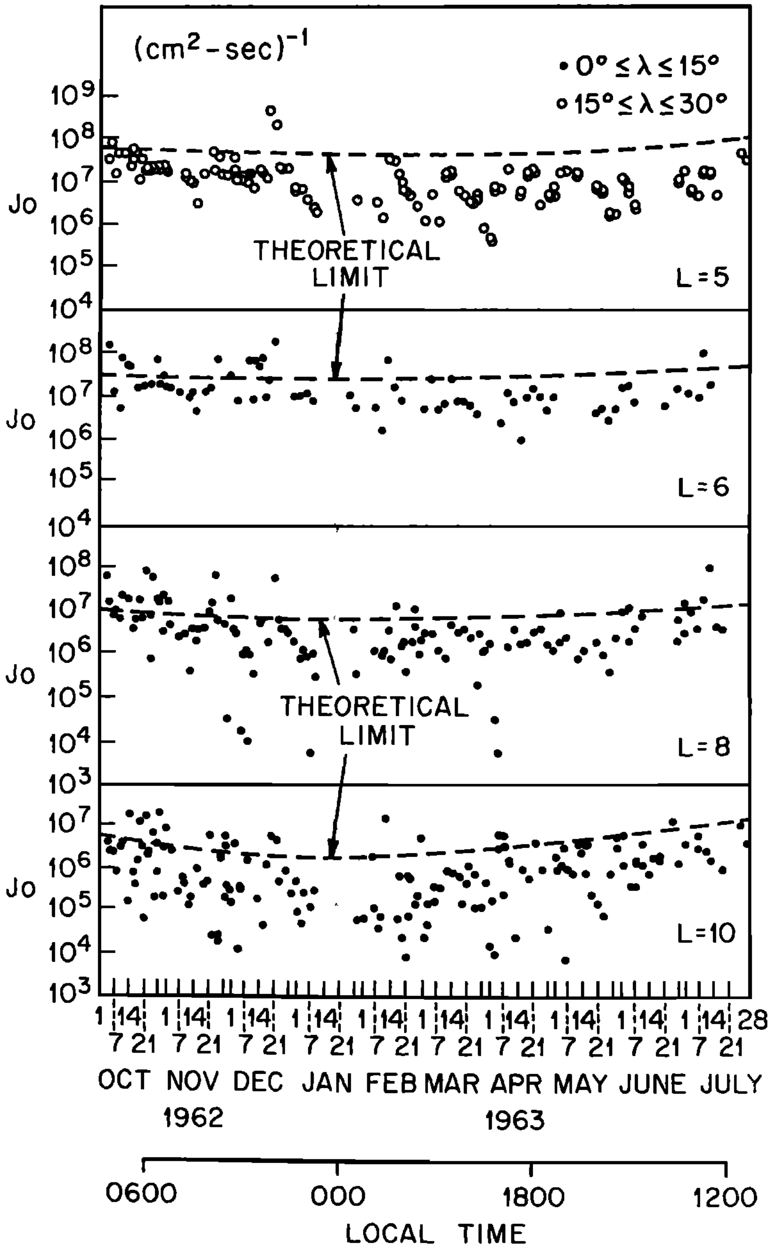


Fig. 12. Limitation on trapped electrons >40 kev as calculated by whistler-wave turbulence (after *Kennel and Petschek* [1966]).

$$E_{Re} = E_B \frac{(T_\parallel/T_\perp)^3}{1 - T_\parallel/T_\perp} \tag{4.4}$$

$$E_{Ri} = E_B \frac{(T_\parallel/T_\perp)^3}{(1 - T_\parallel/T_\perp)^2} \tag{4.5}$$

Now E_B, in units of ev, can be expressed in terms of the magnetic-flux density B_0 in γ and the cold electron density n_{0c} in cm^{-3}

$$E_B(\text{ev}) = \frac{B_0^2(\gamma)}{0.4 n_{0c}(\text{cm}^{-3})} \tag{4.6}$$

Thus, say, for $B_0 \sim 200\ \gamma$, $n_{0c} = 1\ cm^{-3}$, E_B is 100 kev. Therefore, for this example, if the anisotropy, $T_\parallel/T_\perp$, is 0.8, 500-kev electrons and 2500-kev protons resonate with the wave, lose energy, and precipitate through the pitch-angle diffusion.

However, one can see from expression 4.6 that the energy of resonant hot particles can be significantly reduced if n_{0c} is increased, say, by a factor of 10^2. This occurs inside the plasmapause, leading to the expectation that an anomalous loss of particles inside the plasmapause may occur. *Cornwall et al.* [1970] calculated such an effect and showed a good agreement with observation. N. Brice (unpublished data, 1971) proposed an interesting experiment of producing an artificial enhancement of energetic-particle precipitation through injection of cold plasma, based on the dependence shown in equation 4.6.

Recently a good correlation between the whistler noise level and the particle-precipitation rate has been observed by A. L. Vampola, H. C. Koons, and D. A. McPherson (unpublished data, 1971) and *Oliven and Gurnett* [1968], which gives evidence for a loss mechanism due to the cyclotron instability.

Haerendel [1971] extended the calculation of Kennel and Petschek to obtain the limiting flux of protons. *Thorne* [1968] considered an obliquely propagating whistler wave in which the Landau pole ($\omega = k_\parallel v_\parallel$ resonance) contribution becomes important. As was discussed for the electrostatic case (subsection 3.1*a*), $\partial f_0/\partial v$ must be positive at the Landau resonant velocity $v_\parallel = \omega/k_\parallel$ in order to have Re $\sigma < 0$, rather than the anisotropy needed for the cyclotron-resonant velocity. Thorne has shown that an unducted whistler propagating obliquely to the magnetic field may be amplified by the secondary peak ($\partial f_0/\partial v > 0$) in the energy distribution around 10 kev. *Bird and Schmidt* [1969] have considered the effect of loss-cone distribution and two-humped energy distribution on whistler-wave instabilities, but they concluded that these effects are not significant.

We conclude this section by saying that the cyclotron-wave instability serves as an important loss mechanism for radiation-belt particles. However, because a large anisotropy is required, this mechanism is less effective for ring-current particles, except possibly inside the plasmapause.

In addition, scattering due to an electrostatic instability arising from anisotropic temperatures (subsection 3.1*b*), which has not yet been considered, may be important as loss mechanism for the ring current particles. (At the stage of galley proof the author was informed of a work by *Young et al.* [1971].)

4.2. Instability in the Auroral Region

Here we discuss instabilities that are characteristic of auroras within the auroral region. A general review of the role of plasma instabilities in auroral dynamics has been presented in a recent paper [*Hasegawa*, 1971*a*]. Basically two different physical contributions of plasma instabilities have been considered, that which leads to the direct cause of auroral breakup and that which appears in consequence of the highly nonequilibrium situation caused by the breakup. Let us start with the former group.

Explanations of the auroral breakup using plasma instabilities have been made by many authors. *Coppi et al.* [1966] considered the tearing-mode instability of the neutral sheet (subsection 3.2*d*). They pointed out that the acceleration of plasma-sheet electrons due to the instability leads to energetic-electron precipitation to produce the auroral breakup. We will discuss this instability in section 4.4.

Chamberlain [1963], *Swift* [1967*b*], and *D'Angelo* [1969] have considered drift-wave instabilities of various kinds in the ring-current particles as a direct cause of the auroral breakup. However, as was discussed in section 4.1, there are several severe difficulties in satisfying the instability condition. In addition to these, *Cornwall* [1970] pointed out that a high-frequency velocity-space instability that can very easily be generated in the auroral region produces a large velocity-space diffusion, which stabilizes a low-frequency instability such as the drift mode.

Observations of a large field-aligned current during substorms [*Cloutier et al.*, 1970; *Cummings and Dessler*, 1967; *Zmuda et al.*, 1966] have led to a speculation of current-generated instabilities. *Hasegawa* [1970*b*] has applied the helical instability (subsection 3.2*d*) to the auroral sheet-current of aurora to show that the curtain shape of the aurora may be the consequence of this instability. *Swift* [1965] has suspected occurrence of the electron-ion two-stream instability (subsection 3.1*a*), which leads to an anomalous resistivity [*Buneman*, 1958] and the production of a large electric field parallel to the magnetic field. Such an electric field is claimed to cause acceleration of electrons to tens of kev, which could serve as a direct cause of auroral breakup.

However, the Swift theory has two difficulties. One is that the electric field thus produced is directed in such a direction that the cold electrons that carried the original current are accelerated. Because the cold electron reservoir is in the ionosphere, the accelerated electrons should be directed upward along the field lines, which is opposite to what is observed during auroral breakup. Another difficulty is satisfying the instability condition. Because in the ionosphere $T_i \sim T_e$, the threshold velocity of electrons for the electron-ion two-stream instability is the electron thermal speed v_{Te} (subsection 3.1*a*), rather than the ion sound speed $c_S[= v_{Te}(m_e/m_i)^{1/2}]$. Thus it requires a rather large current. In fact *Ossakow* [1968] has shown that the condition of instability is marginal at the upper ionosphere. In view of this fact, *Kindel and Kennel* [1971] looked at the electrostatic ion-cyclotron instability produced by streaming electrons, and showed a lower threshold. (Such an instability can be obtained by using the dispersion relation for the electrostatic mode propagating obliquely to the magnetic field (equation 3.27) and by assuming suitable distribution functions for electrons and protons.) However, there still remains the difficulty, mentioned above, of the direction of the electric field.

We conclude that attempts at an explanation of auroral breakup using plasma instabilities have not yet been successful.

Now the fact that energetic-particle precipitation exists during the auroral breakup leads us to suspect that instabilities may be produced by those particles. *Nishida* [1964] considered the excitation of the shear Alfvén mode by precipitating electrons (subsection 3.1*c*) to explain micropulsations associated with auroral breakup. *Cornwall* [1965] considered similar excitation by precipitating protons to explain emissions in the ULF to VLF frequency ranges. *Coroniti and Kennel* 1970*b*] considered the drift wave mode (subsection 3.2*a*) excited by the large temperature gradient at the inner edge of the plasma sheet, to explain micropulsations in auroras having periods of 5–15 seconds.

Perkins [1968] has considered an electrostatic instability propagating obliquely to the magnetic field, produced by a monoenergetic electron distribution in a cold plasma background. He claims that stochastic acceleration by this mode can easily produce 10-kev electrons.

Quite uniquely, *Hallinan and Davis* [1971] have attributed curls of approximately 10-km size, produced in the aurora sheet, to the Kelvin-Helmholtz instability as discussed in subsection 3.2*c*, and concluded from the direction of the curl formation that the sheet is electron-rich (cf. Figure 13).

That the auroral striation could be a consequence on the **E** × **B** instability (subsection 3.1*e*) has been suspected by Linson in the context of a paper on plasma-cloud instability [*Linson and Workman,* 1970]. We conclude that phenomena that accompany the aurora may have relevant explanations in terms of plasma instabilities.

4.3. Instability of Magnetospheric Tail

As was shown in subsection 3.2*d*, an infinitely extended sheet-current pinch becomes unstable for a long wavelength perturbation ($\lambda \gg a$, where a is the thickness of the current sheet) when, and only when, a finite resistivity exists in the current. The instability is called the tearing-mode instability, and the sheet current is torn into a number of segments by the magnetic field, as shown in Figure 14. When one applies the instability to the neutral sheet in the magnetospheric tail, the immediate difficulty is the finite resistivity needed for the instability. *Hoh* [1966*a*] has shown that electron Landau damping can contribute to the resistivity, and that the instability can occur in the absence of collisional resistivity. *Coppi et al.* [1966], using the same idea, have explicitly calculated the growth time (time needed to exponentiate the perturbation) for the parameters in the actual magnetospheric tail and have shown that it is of the order of 10 seconds.

However, two difficulties in this idea have been pointed out by *Laval and Pellat* [1968] and by *Hoh and Bers* [1966]. According to Laval and Pellat, the instability is stabilized when the anisotropy in temperature is taken into account.

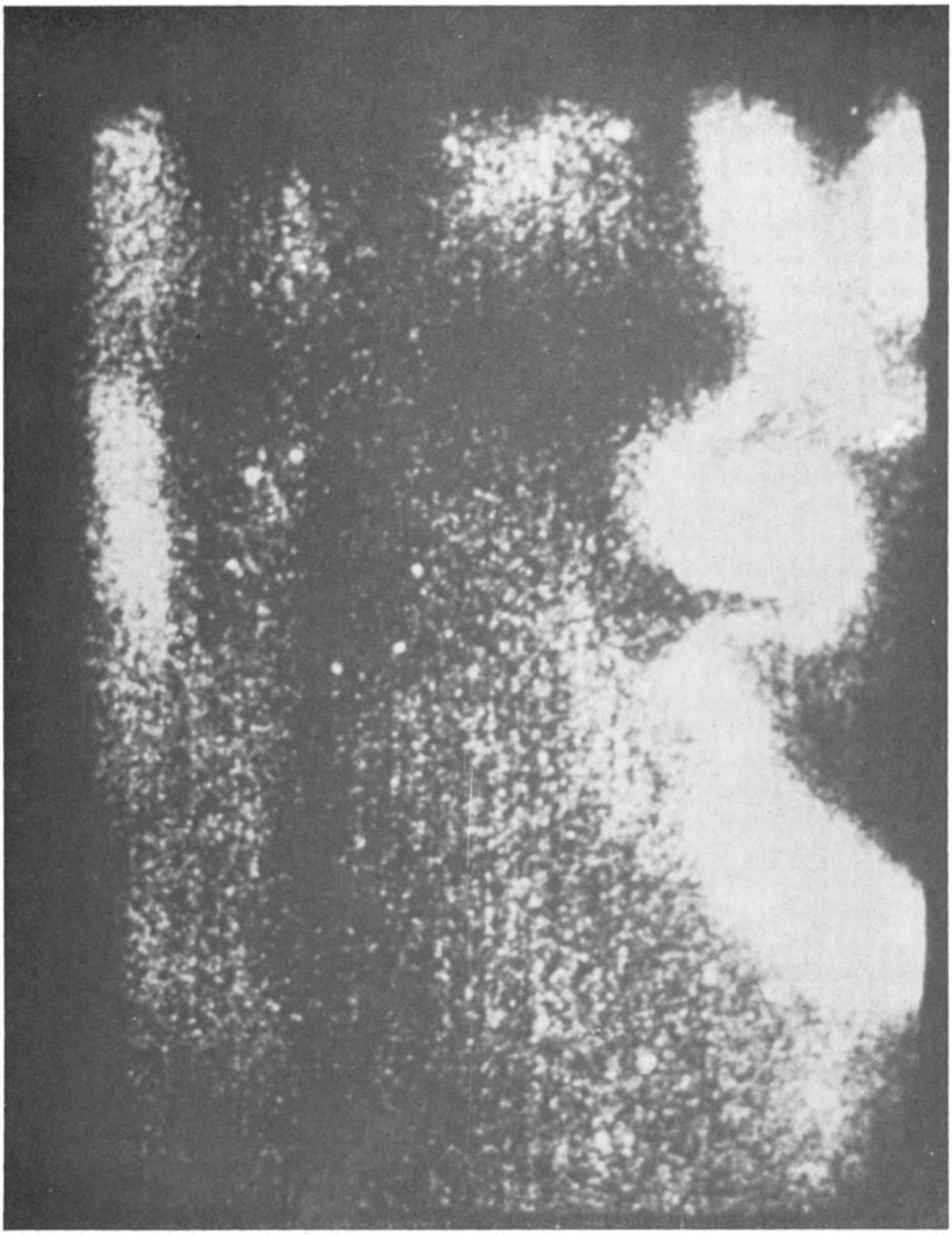


Fig. 13. Observation of Kelvin-Helmholtz instability in an auroral sheet (after *Hallinan and Davis* [1971]).

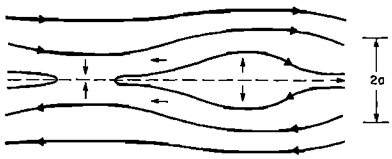


Fig. 14. Tearing-mode instability of the neutral sheet in the magnetospheric tail.

That is, the collisionless tearing mode is stable when

$$1 - \frac{T_\perp}{T_\parallel} > \frac{\rho_e}{a} \tag{4.7}$$

where ρ_e is the electron cyclotron radius. Because $\rho_e/a \ll 1$, equation 4.7 implies that the mode is stable essentially when $T_\perp < T_\parallel$. Hoh and Bers have calculated the number of electrons that contribute to the resonant interaction through Landau damping and, by considering the resultant source of free energy, they have obtained the maximum amplitude of the field perturbations. For a reasonable choice of parameters, they have shown that the maximum amplitude of the magnetic-field perturbation produced by the instability is of the order of or smaller than 0.5 γ, which is much smaller than the average tail field ($\sim$15 γ).

In view of these arguments, the use of the collisionless tearing-mode instability for gross tail dynamics, or to cause magnetic substorms, seems rather difficult. However, as was mentioned by *Brice* [1970], if the current in the neutral sheet becomes so large that a two-stream-type velocity space instability occurs, it will provide a large enough anomalous resistivity to dissipate field energy very easily. Such an instability may very well trigger the tearing-mode instability. This seems to be an extremely interesting proposal, especially because it may be able to explain the explosive phase of the auroral substorm.

4.4. Instability in the Ionosphere

In a partially ionized plasma, because of the difference in $\omega_c\tau$, where τ is the mean-free time, $\mathbf{E}_0 \times \mathbf{B}_0$ drift can produce a two-stream situation between electrons ($\omega_{ce}\tau_e \gg 1$) and ions ($\omega_{ci}\tau_i \lesssim 1$) (subsection 3.1*e*). *Buneman* [1963] has shown that an instability produced by two such streams can excite a field aligned sound wave propagating in the $\mathbf{E}_0 \times \mathbf{B}_0$ direction in the ionosphere. His idea has been extended by *Farley* [1963] using the Vlasov equation to obtain the critical drift velocity, when temperature effects are not negligible. He has shown that the threshold drift velocity of the electrons is $\sim v_{Ti} \sim 0.1 v_{Te}$, if $T_i \sim T_e$. It is interesting to see that the threshold can become lower than that of the ion sound-wave instability propagating parallel to the magnetic field. This is primarily because ion Landau damping is absent for wave propagation normal to the magnetic field.

As was mentioned in subsection 3.1*e*, if one combines the effects of a density gradient ∇n_0 and a dc electric field $\mathbf{E}_0$, the instability condition can be more easily satisfied if $\mathbf{E}_0 \cdot \nabla n_0 > 0$. This is particularly important when electrons are not collision-free and their drift velocity is smaller than the thermal velocity. Tsuda and Sato (See *Tsuda et al.* [1969] for previous references) have applied this idea to explain equatorial *E*-layer irregularities. *Reid* [1969] has extended works by Tsuda and Sato, including the altitude-variation of the neutral atmosphere as well as of ionization, and has applied them to the *F*-region irregularities.

Cunnold [1969] has shown that by considering only the effect of the density gradient (drift-wave instability) the *F*-layer irregularity can be explained.

The existence of a dc electric field, as well as a density gradient in electrons, is a well-known fact. Other parameters such as number density, collision frequency, and magnetic-field intensity, are also well-known. Thus if instability condition using these parameters is satisfied, it is undeniable evidence. We conclude that the production of ionospheric irregularities is very likely the consequence of those instabilities.

5. GENERATION OF WAVES AND PULSATIONS

We discuss here various mechanisms of wave generation by plasma instabilities. We divide the chapter into four sections: instabilities of electron-cyclotron waves, proton-cyclotron waves, other instabilities proposed for excitation of pulsations, and wave generation in the magnetosphere. The first three apply to pulsations and VLF emissions primarily observable on the ground, whereas the last applies to waves in the magnetosphere observable only by satellites.

5.1. Excitation of Electron-Cyclotron Waves (Whistlers)

That a whistler wave may be generated by a beam of particles was first pointed out in a paper by *Kimura* [1961], in which he showed that the whistler wave could be amplified by a proton beam (subsection 3.1*c*). The idea of wave generation by a beam has since become so popular that there has been a tendency to attribute any wave observation to some kind of beam with a convenient energy.

As was shown in subsection 3.1*c*, in an infinite medium the two-stream cyclotron-wave instability occurs between different species. Namely, the electron-cyclotron wave can be excited only by the proton beam or vice versa. *Bell and Buneman* [1964] have shown, however, that when the beam is gyrating, interaction between the same species may be possible; they gave as an example whistler-wave amplification using a gyrating-electron beam.

The idea presented by *Weibel* [1959], which predicts an unstable electromagnetic mode for anisotropic temperature, as discussed in subsection 3.1*c*, has been picked up by *Kennel* [1966] to be applied as an additional cause of whistler amplification. He also pointed out the importance of the Landau pole ($\omega = k_{\parallel}v_{\parallel}$) to the obliquely propagating whistlers and extended this idea to show the stabilization of unducted whistlers due to the Landau damping by low-energy to thermal electrons [*Kennel and Thorne*, 1967].

Very recently *Lee and Crawford* [1970] have pointed out that the cyclotron-wave instability due to the anisotropic temperature is not always convective (spatially amplifying) as was assumed by *Kennel and Petschek* [1966] in their famous paper on calculation of the limiting fluxes.

Hruška [1966] and *Liemohn* [1968] considered the two-stream cyclotron-wave instability including the effect of anisotropic temperature. *Das* [1967] discussed the effect of a loss-cone distribution on whistler-wave amplification.

In conclusion, whistler waves may be amplified by a proton beam, by a gyrating-electron beam, or by the anisotropic temperature of electrons. When a wave propagates obliquely with respect to the magnetic field, it can either be Landau-damped or amplified, depending on the energy distribution of electrons in the magnetic field direction.

5.2. Excitation of Proton-Cyclotron and Alfvén Waves

As far as the nature of the instability is concerned, the excitation mechanism of the proton-cyclotron mode is essentially the same as that of the electron-cyclotron wave. The existence of an electron beam, a gyrating-proton beam, or an anisotropic proton temperature, can cause the instability.

Observationally, however, the corresponding frequency lies naturally much lower (~1 Hz or lower) than the whistler case. *Jacobs and Higuchi* [1969] solved the dispersion relation of the plasma-to-hot-proton-beam interaction to obtain a growing-wave solution in the Pc 1 range. *Criswell* [1969] discussed the morphology of Pc 1 pulsations based on a similar instability. There are many other works published on related problems. References can be found in the most recent papers on this topic.

In addition to the interaction around the proton-cyclotron frequency, as was shown in Figure 5, there is another frequency range where a beam can generate an instability. This instability has such an interesting nature that the excited wave has a frequency proportional to the unneutralized-charge density. In addition, the polarization of the wave depends on the sign of the excess charge; a left- (right-) hand polarized wave is excited by an electron- (proton-) rich beam.

These instabilities have been applied by *Nishida* [1964] to explain irregular magnetic micropulsations and by *Kimura and Matsumoto* [1968] to explain Pc 5 micropulsations.

5.3. Other Theories of Instability-Generated Pulsations

Although most of the theories of instability-generated pulsations are based on excitations of a proton-cyclotron wave or an Alfvén wave by a beam or by anisotropic temperature, there are some other examples. Swift published two papers of this nature. One explains the long-period micropulsations (~1-min period) by an electrostatic drift wave instability [*Swift,* 1967*b*], such as the one discussed in subsection 3.2*a*. As explained in this text, the electrostatic drift instability is rather unlikely to occur in the magnetosphere because of the ion Landau damping in a high β situation, and because of cold-electron short-circuiting. Besides, the theory as it stands is developed purely for electrostatic perturbations; hence a coupling scheme to the electromagnetic mode must be worked out to compare it with the observations of pulsations in the geomagnetic field.

The second paper applies the loss-cone instability found by *Post and Rosenbluth* [1966], to explain VLF chorus [*Swift,* 1968]. When there is a loss-cone distribution (a distribution in which particles with pitch angles less than the loss-cone angle are absent), the dielectric constant becomes active (Im $\epsilon < 0$) for waves propagating almost perpendicular to the magnetic field. Instability occurs when this ion mode couples with the cold-electron dielectric constant, and waves of the ion Bernstein mode, at multiples of the proton-cyclotron frequency, are excited. The wave is stabilized, however, when the electron temperature is high because of Landau damping.

Swift assumed a rather large loss-cone angle (45°) to show the possibility of the instability. Since the excited wave is again electrostatic, a more careful study, including realistic loss-cone distributions and coupling to the electromagnetic mode, has to be made.

Coroniti and Kennel [1970*a*] consider modulations of the energetic-electron precipitation by low-frequency micropulsations. Using an idealized model of critical whistler-wave turbulence, they have shown that the precipitation modulation depends exponentially on micropulsation amplitude, when the micropulsation period is less than the electron-precipitation lifetime. This is primarily because the growth rate of the whistler-wave instability due to anisotropic temperature is proportional to B_0, as shown in equation 3.58. Hence if B_0 changes slowly with time, so does the growth rate γ, and the precipitation that has exponential dependency on $\gamma(\sim e^{\gamma t})$ is strongly modulated by a small change in B_0.

5.4. Wave Generation in the Magnetosphere

Recent satellite measurements have made it possible to detect directly wave phenomena in the deep magnetosphere [*Brown et al.*, 1968; *Dunckel and Helliwell,* 1969; *Gurnett et al.*, 1969; *Russell and Holzer,* 1970; *Scarf et al.*, 1968, 1970; *Cummings et al.*, 1970].

These observations have revealed that, in addition to expected whistler-related phenomena, there are low-frequency waves (period of a few minutes [*Russell and Holzer,* 1970]), high-frequency waves (a few kHz [*Scarf et al.*, 1970]), and even electrostatic waves (a few kHz [*Kennel,* 1970]), which are confined around equatorial regions in the magnetosphere.

Dungey and Southwood [1970] have tried to explain the low-frequency waves by bouncing protons in the geomagnetic-mirror field. *Kennel et al.* [1970] tried to explain the electrostatic mode in terms of the known electrostatic instability due to anisotropic temperature (subsection 3.1*b*). These wave phenomena are quite intriguing, and many problems are yet unsolved. Careful theoretical study should produce many fruitful results.

There is one rather clear example in which a plasma instability has caused a wave phemonenon in both magnetic field and particle fluxes. This is the case reported by *Brown et al.* [1968] and interpreted by the author [*Hasegawa,* 1969*a*]. The observed oscillation is reproduced in Figure 15.

The observation was made by Explorer 26, which was located at 5.11 R_E geocentric distance, 1400 LT, and latitude of 17°. From this figure can be seen (*a*) a strong diamagnetic effect (indicating $\beta \sim 1$) starting at 6:15 and suddenly terminating at 6:20, shown by point A; (*b*) a large anisotropy of the proton fluxes ($\beta_\perp > \beta_\parallel$); and (*c*) the start of large out-of-phase oscillations of the field and the fluxes shortly after the point A.

The first two facts strongly indicate the occurrence of a mirror instability (subsection 3.1*d*). When such an instability is set up, it can terminate a further increase in anisotropy of the proton flux and trigger the succeeding oscillations. In fact, by studying the proton data provided by L. Davis of GSFC, it was found

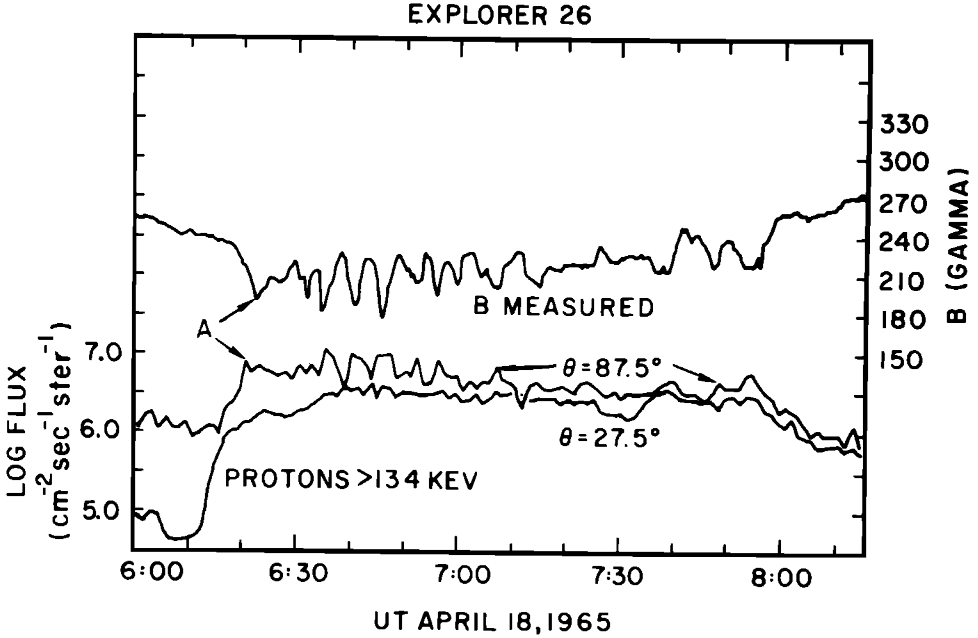


Fig. 15. Observed variations of proton fluxes and geomagnetic field at ~5 R_E (after *Brown et al.* [1968]).

that near point A in Figure 15, $\beta_\perp/\beta_\parallel \sim 2$ and $\beta_\perp \sim 1$, which satisfies the condition of instability shown in equation 3.72.

Because the mode causing the mirror instability is nonoscillatory (equation 3.70), other effects are required to explain the oscillations clearly observed after 6:30. It is suggested that the oscillation is produced by coupling with the drift wave created by the ion drift perpendicular to the magnetic field. This will give rise to a real frequency $\omega \sim k_\perp v_d$, where v_d is the proton diamagnetic drift-speed, and the perpendicular wave number $k_\perp$ may be chosen to be $\omega_{ci}/\langle v_\perp \rangle_i$, corresponding to its value for the maximum growth rate. Observed values of average proton energy (~20 kev) and the magnetic-field strength (~200 γ) give a frequency that is in good agreement with the observed frequency.

6. CONCLUDING REMARKS

Various theories of plasma instabilities and their applications to magnetospheric dynamics have been presented. Possible instabilities that have been proposed in the literature are summarized in Figure 16. Not all of these are realistic, as has been discussed in chapters 4 and 5. We did not discuss the stability of the magnetospheric boundaries, primarily because of the lack of equilibrium there [*Lerche*, 1967; *Parker*, 1967*a*, *b*; 1968*a*, *b*], but this is by no means a less important area. One possible solution for this problem, suggested by *Aviatar and Wolf* [1968] is to use the idea of 'turbulent equilibrium' excited by a two-stream instability. In any case, a direct application of the Kelvin-Helmholtz instability without taking these points into account seems meaningless.

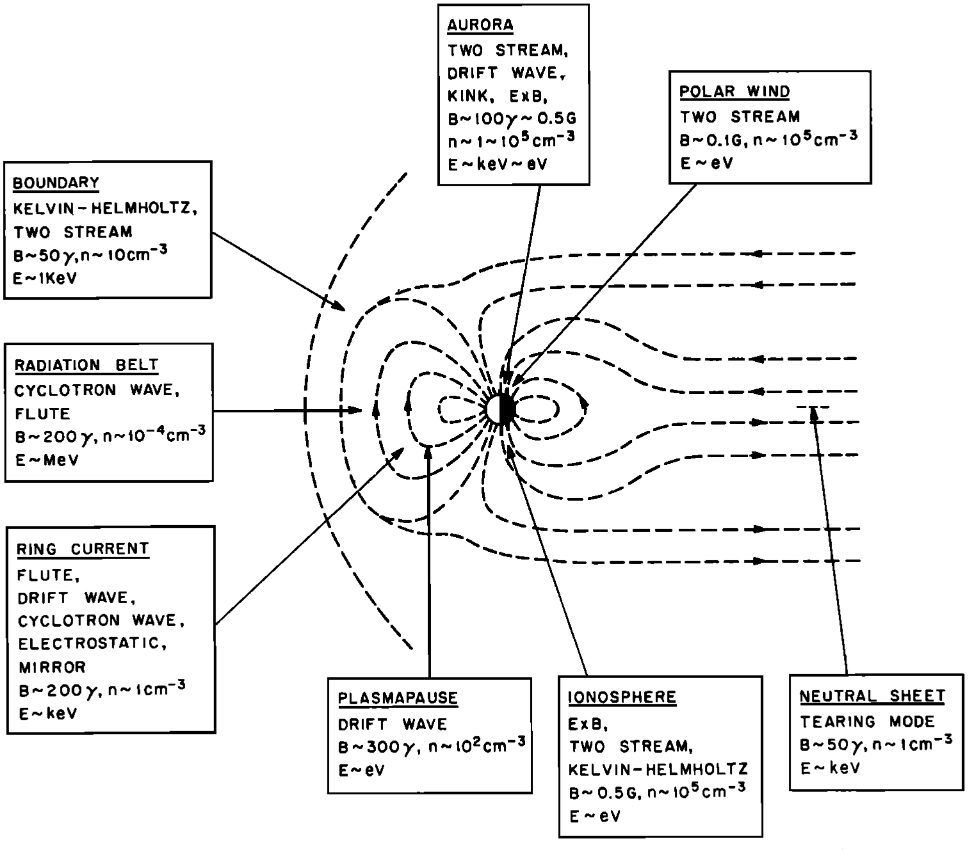


Fig. 16. Plasma instabilities that have been predicted in various portions of the magnetosphere.

As far as plasma instability is concerned, several important problems remain that should be worked out in the immediate future. They are (1) the drift-wave instability in a high β situation, especially that of compressional mode; (2) the electrostatic instabilities and their consequence to particle scattering; (3) a more careful treatment of the tearing mode instability, including the possible anomalous resistivity produced by the neutral-sheet current; (4) modification of drift modes, including the bounce-resonance effects of particles; (5) a realistic treatment of magnetospheric boundaries; and (6) several hidden instabilities during magnetospheric substorms.

In addition to direct dynamic changes in the magnetosphere caused by coordinate-space instabilities, wave generation by velocity-space instabilities produces various important consequences to magnetospheric particles, such as diffusion [*Birmingham et al.*, 1967; *Fälthammer*, 1970; *Lanzerotti et al.*, 1970; *Nakada and Mead*, 1965; *Newkirk and Walt*, 1968], heating and acceleration [*Barnes*, 1967; *Hasegawa*, 1969*b*; and *Laval and Pellat*, 1970], and particle scattering [*Cornwall*, 1964; *Dungey*, 1964; *Roberts*, 1966; *Aviatar*, 1967; *Roberts and Schulz*, 1968].

This paper can be concluded by saying, without exaggeration, that plasma instabilities control the dynamics of the magnetosphere.

APPENDIX A. LIST OF NOTATIONS

n = number density.
v_0 = stream velocity.
v_p $(= \omega/k)$ = phase velocity.
v_T $(= (\langle v^2 \rangle)^{1/2}$ = thermal velocity.
$\langle \ \rangle$ = unstable average.
Subscript i = ion (proton).
Subscript e = electron.
Subscript s = stream.
Subscript 0 = unperturbed (dc) quantities.
Subscript 1 = perturbed (ac) quantities.
Subscript $\perp$ = perpendicular to the magnetic field.
Subscript $\|$ = parallel to the magnetic field.
E = electric field intensity.
B = magnetic flux density.
m = mass of a particle.
J = current density.
ω_p $(= (q^2 n_0/\epsilon_0 m)^{1/2})$ = plasma frequency.
ω_c $(= qB_0/m)$ = cyclotron frequency.
ϵ_0 = vacuum dielectric constant = 8.854×10^{-12} Farad/meter.
W = energy density of a wave.
σ = equivalent plasma conductivity.
ϵ = equivalent plasma dielectric constant.
v_g $(= \partial\omega/\partial k)$ = group velocity.
φ = electrostatic potential.
Im ω = imaginary part of ω.
Re ω = real part of ω.
β $(=nkT/(B^2/2\mu_0))$ = ratio of plasma pressure to the magnetic field pressure.
f = distribution function.
c_s $(= v_{Te}(m_e/m_i)^{1/2})$ = ion sound speed.
$v_\perp$ $[= (v_x^2 + v_y^2)^{1/2}]$ = perpendicular velocity in cylindrical coordinate system with $z \| \mathbf{B}$.
$v_\|$ $[= v_z]$ = parallel velocity in cylindrical coordinate with $z \| \mathbf{B}$.
P = principal value of integral.
c = speed of light in vacuum = $2.99 \cdots \times 10^8$ m/sec.
$v_A (= \omega_{ci} c/\omega_{pi})$ = Alfvén speed.
Z = plasma dispersion function, equation 3.29.
μ_0 = vacuum permeability = 1.26×10^{-6} Henry/meter.
μ $(= m\langle v_\perp^2 \rangle/2B_0)$ = magnetic moment.
ν = collision frequency.
μ $(= e/\nu m)$ = mobility.
D = diffusion constant.
v_d $(= \kappa v_T^2/\omega_c)$ = diamagnetic drift velocity.

$\mathbf{v}_g \left(= -\frac{\mu \nabla(B_0) \times \mathbf{B}_0}{q B_0^2} \right)$ = gradient B drift velocity.

$\kappa \left(= -\left| \frac{d \ln n_0}{dx} \right| \right)$ = measure of density gradient.

$\mathbf{v}_R \left(= -\frac{m\langle v_\parallel^2 \rangle \mathbf{R} \times \mathbf{B}_0}{q R^2 B_0^2} \right)$ = curvature drift velocity where $\mathbf{R}$ = radius of curvature defined by $\frac{\mathbf{R}}{R^2} = \left(\frac{\mathbf{B}}{B} \cdot \nabla \right) \frac{\mathbf{B}}{B}$.

$\rho\ (= v_{T\perp}/\omega_c)$ = cyclotron radius.

η = resistivity.

Acknowledgments. The author wishes to thank L. J. Lanzerotti, C. Maclennan, and W. L. Brown for reading and criticizing of the manuscript. He also appreciates discussions with H. Furth on the tearing-mode instability and with L. M. Linson on the Kelvin-Helmholtz instability.

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(Received March 29, 1971; revised May 6, 1971.)